A Bayesian Lower Bound for Parameter Estimation of Poisson Data Including Multiple Changes (extended)

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Abstract

This paper derives lower bounds for the mean square errors of parameter estimators in the case of Poisson distributed data subjected to multiple abrupt changes. Since both change locations (discrete parameters) and parameters of the Poisson distribution (continuous parameters) are unknown, it is appropriate to consider a mixed Cramér-Rao/Weiss-Weinstein bound for which we derive closed-form expressions and illustrate its tightness by numerical simulations.

I. Introduction

Parameter estimation in the context of discrete Poisson time series submitted to multiple abrupt changes is of practical interest in many applications, such as the segmentation of multivariate astronomical time series [1]–[3]. In this context, the observed data (distributed according to a Poisson distribution) are subjected to abrupt changes whose locations are unknown. The values of the Poisson parameters associated with each interval are also unknown quantities that need to be estimated. Several strategies have been investigated in the literature [4]–[8]. However, to the best of our knowledge, lower bounds for the mean square error of the resulting simulators have been derived only in a few specific cases. For instance, the case of a single change-point in the observation window was studied in [9]. The case of multiple changes was considered in [10]. The difficulty of deriving bounds for the parameters of piece-wise stationary data is mainly due to the discrete nature of changepoint locations for which classical bounds such as the Cramér-Rao bound (CRB) are not appropriate anymore. In [9], [10], the authors considered a lower bound for the mean square error (MSE) that does not require the differentiability of the log-likelihood function. Specifically, deterministic bounds, such as the Chapman-Robbins bound, have been derived for a single change-point in [9], and then extended to multiple changes in [10], with the strong assumptions that the Poisson parameters are known. On the other hand, and in order to improve the tightness of the resulting bound, we proposed in [11], [12] the use of the Weiss-Weinstein bound (WWB), which is known to be one of the tightest bound in the family of the Bayesian bounds. Nevertheless, these analyses were limited to the case of known Poisson parameters both in the single [11] and multiple [12] changepoint scenarios.

In this paper, we fill this gap by proposing and deriving a new Bayesian lower bound for the global MSE (GMSE) of the parameters of Poisson distributed data subjected to multiple changepoints. The proposed bound is a mixed Cramér-Rao (CR)/Weiss-Weinstein (WW) bound adapted to the fact that some unknown parameters are discrete (the WW part of the mixed bound is associated with the change locations) and that the other parameters are continuous (the CR part of the mixed bound is associated with the Poisson parameters). The idea of combining these two bounds had already been introduced in [13], under a recursive form. Of course, using a WWB for both discrete and continuous parameters would be theoretically possible. However, the WW bound is expressed as the supremum of a set of matrices whose computation is infeasible in our scenario. Thus, the mixed Cramér-Rao/Weiss-Weinstein bound is the appropriate alternative, whose computation can be achieved using a convex optimization procedure to compute this supremum based on the computation of the minimum volume ellipsoid covering a union of derived ellipsoids.

II. Multiple Change-Points in Poisson Time-Series: problem formulation

We consider an independent discrete Poisson time series subjected to multiple changes. The resulting observation vector $\boldsymbol{x} = [x_1, \ldots, x_T]^T$ (of length T) is defined as

$$\begin{cases} x_t \sim \mathcal{P}(\lambda_1), & \text{for } t = 1, \ldots, t_1 \\ x_t \sim \mathcal{P}(\lambda_2), & \text{for } t = t_1 + 1, \ldots, t_2 \\ \quad \vdots & \quad \vdots \\ x_t \sim \mathcal{P}(\lambda_{K+1}), & \text{for } t = t_K + 1, \ldots, T, \end{cases} \tag{1}$$

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where $\mathcal{P}(\lambda_k)$, for $k = 1, \ldots, K+1$, denotes the Poisson distribution of parameter λ_k on the k-th segment, i.e., $\Pr(x_t = \kappa_t) = \lambda_k^{\kappa_t} \exp\{-\lambda_k\}/(\kappa_t!)$, K denotes the total number of changes (assumed to be known), and t_k denotes the k-th change location, i.e., the sample point after which the parameter λ_k of the current segment switches to λ_{k+1}. The segmentation problem addressed in this work consists of i) segmenting the time series $\boldsymbol{x}$, i.e., estimating the locations of the changes t_k, and ii) estimating the Poisson parameters λ_k on each segment. The resulting unknown parameter vector is $\boldsymbol{\theta} = \left[\boldsymbol{\lambda}^T, \boldsymbol{t}^T\right]^T$, with $\boldsymbol{\lambda} \triangleq [\lambda_1, \ldots, \lambda_{K+1}]^T$ and $\boldsymbol{t} \triangleq [t_1, \ldots, t_K]^T$. This unknown parameter vector lies in the parameter space $\Theta = \mathbb{R}_+^{K+1} \times \{1, \ldots, T\}^K$, where $\mathbb{R}_+$ denotes the set of real positive numbers. Using a Bayesian framework, we consider that both vectors $\boldsymbol{\lambda}$ and $\boldsymbol{t}$ are assigned a known prior. More precisely, the Poisson parameters λ_k are assumed to be independent and identically distributed (i.i.d.), and are assigned the conjugate gamma distributions with parameters α_k and β, leading to the following prior

$$f(\boldsymbol{\lambda}) = \prod_{k=1}^{K+1} \frac{\beta^{\alpha_k}}{\Gamma(\alpha_k)} \lambda_k^{\alpha_k - 1} \exp(-\beta\lambda_k) \, \mathbb{I}_{\mathbb{R}_+}(\lambda_k) \tag{2}$$

in which $\mathbb{I}_{\mathcal{E}}(.)$ denotes the indicator function on the set $\mathcal{E}$, and $\Gamma(.)$ denotes the usual gamma function, i.e., for $\alpha > 0$, $\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} \exp(-x) \mathrm{d}x$. On the other hand, we assume that each change location t_k, for $k = 1, \ldots, K$, is defined as the following random walk $t_k = t_{k-1} + \varepsilon_k$ where ε_k are i.i.d. variables following a discrete uniform distribution on the set of integers $\{1, \ldots, \tau\}$, and $t_0 = 0$. The value of τ is chosen so that the final change t_K is at least located before the last observation, i.e., the maximum possible value for τ is $\tau_{\max} = \lfloor (T-1)/K \rfloor$, where $\lfloor . \rfloor$ denotes the floor function. Consequently, we obtain the following prior distribution for the unknown vector $\boldsymbol{t}$

$$\Pr(\boldsymbol{t} = \boldsymbol{\ell}) = \frac{1}{\tau^K} \prod_{k=1}^{K} \mathbb{I}_{\{\ell_{k-1}+1, \ldots, \ell_{k-1}+\tau\}}(\ell_k) \tag{3}$$

with $\ell_0 = 0$. Since vectors $\boldsymbol{\lambda}$ and $\boldsymbol{t}$ are independent, the joint prior for $\boldsymbol{\lambda}$ and $\boldsymbol{t}$ is expressed as $f(\boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell}) = f(\boldsymbol{\lambda}) \Pr(\boldsymbol{t} = \boldsymbol{\ell})$.

From the model (1), the likelihood of the observations can be written as

$$f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell}) = \prod_{k=1}^{K+1} \prod_{t=\ell_{k-1}+1}^{\ell_k} \frac{\lambda_k^{\kappa_t}}{\kappa_t!} \exp\{-\lambda_k\}. \tag{4}$$

The aim of the present paper is to study the estimation performance of the vector $\boldsymbol{\theta}$ by deriving a lower bound on the mean square error (MSE) of any Bayesian estimator $\hat{\boldsymbol{\theta}}(\boldsymbol{x})$ of $\boldsymbol{\theta}$. Both subvectors $\boldsymbol{\lambda}$ and $\boldsymbol{t}$ of $\boldsymbol{\theta}$ have to be estimated simultaneously. However, as already mentioned in Section I, the Cramér-Rao bound is not suited for changepoint analysis, since $\boldsymbol{\ell}$ is a vector of discrete parameters. Thus, the idea is to use two different lower bounds w.r.t. each subvector of $\boldsymbol{\theta}$, resulting in a "mixed" Bayesian bound that corresponds to the Bayesian CR bound for the first subvector $\boldsymbol{\lambda}$ of $\boldsymbol{\theta}$, and that corresponds to the so-called WW bound for the second subvector $\boldsymbol{t}$ of $\boldsymbol{\theta}$. As already mentioned, the use of such a combined CR/WW lower bound was initiated in [13] in a target tracking context. Since our framework is different, the next section is devoted to the presentation of this bound, which we will refer to as the "Bayesian Cramér-Rao/Weiss-Weinstein bound" (BCRWWB). It is in fact a special case of a general family of lower bounds exposed in [14].

III. Bayesian Cramér-Rao/Weiss-Weinstein bound

We are interested in studying the estimation performance of a parameter vector $\boldsymbol{\theta}$ that lies in a parameter space $\Theta = \mathbb{R}^{K+1} \times \mathbb{N}^K$. As explained in the previous section, this parameter vector can be split into two subvectors, $\boldsymbol{\lambda} \in \Theta_{\boldsymbol{\lambda}} = \mathbb{R}_+^{K+1}$ and $\boldsymbol{t} \in \Theta_{\boldsymbol{t}} = \mathbb{N}^K$, so that $\boldsymbol{\theta} = \left[\boldsymbol{\lambda}^T, \boldsymbol{t}^T\right]^T$ and $\Theta = \Theta_{\boldsymbol{\lambda}} \times \Theta_{\boldsymbol{t}}$. From a set of observations $\boldsymbol{x} \in \Omega$, the vector $\boldsymbol{\theta}$ can be estimated by using any Bayesian estimation scheme, leading to an estimator $\hat{\boldsymbol{\theta}}(\boldsymbol{x}) = \left[\hat{\boldsymbol{\lambda}}(\boldsymbol{x})^T, \hat{\boldsymbol{t}}(\boldsymbol{x})^T\right]^T$. Let us recall that we aim at obtaining a lower bound on the *global* mean square error (GMSE) of this estimator, which corresponds to the Bayesian CR bound w.r.t. $\boldsymbol{\lambda}$, and which corresponds to the WW bound w.r.t. $\boldsymbol{t}$. The GMSE of $\hat{\boldsymbol{\theta}}(\boldsymbol{x})$ is defined as the $(2K+1) \times (2K+1)$ matrix

$$\mathbf{GMSE}(\hat{\boldsymbol{\theta}}) = \mathbb{E}_{\boldsymbol{x}, \boldsymbol{\theta}} \left\{ \left[\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(\boldsymbol{x})\right] \left[\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(\boldsymbol{x})\right]^T \right\} \tag{5}$$

in which $\mathbb{E}_{\boldsymbol{x}, \boldsymbol{\theta}}\{.\}$ denotes the expectation operation w.r.t. the joint distribution $f(\boldsymbol{x}, \boldsymbol{\theta})$ which depends on both the observations and the parameters. Based on [14], by appropriately choosing some real-valued measurable functions $\psi_k(\boldsymbol{x}, \boldsymbol{\theta})$, $k = 1, \ldots, 2K+1$, defined on $\Omega \times \Theta$ such that the following integrals exist and satisfy $\int_\Theta \psi_k(\boldsymbol{x}, \boldsymbol{\theta}) f(\boldsymbol{x}, \boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} = 0$ for almost every (a.e.) $\boldsymbol{x} \in \Omega$ and for $k = 1, \ldots, 2K+1$, the following matrix inequality holds

$$\mathbf{GMSE}(\hat{\boldsymbol{\theta}}) \succeq \boldsymbol{V} \boldsymbol{P}^{-1} \boldsymbol{V}^T \tag{6}$$

in which $\boldsymbol{V}$ is a $(2K+1) \times (2K+1)$ matrix whose elements are given by

$$[\boldsymbol{V}]_{k,l} = \mathbb{E}_{\boldsymbol{x}, \boldsymbol{\theta}} \{\theta_k \psi_l(\boldsymbol{x}, \boldsymbol{\theta})\} \tag{7}$$

and $\boldsymbol{P}$ is a $(2K+1)\times(2K+1)$ symmetric matrix, whose elements are given by

$$[\boldsymbol{P}]_{k,l} = \mathbb{E}_{\boldsymbol{x},\boldsymbol{\theta}}\left\{\psi_k(\boldsymbol{x},\boldsymbol{\theta})\,\psi_l(\boldsymbol{x},\boldsymbol{\theta})\right\}. \tag{8}$$

Note that the matrix inequality (6) means that the difference between its left and its right hand sides is a nonnegative definite matrix. One key point in the theory developed in [14] is the choice of the measurable functions ψ_k. For k restricted to $\{1,\ldots,K+1\}$ (continuous Poisson parameters), we define these functions as for the CR bound, i.e.,

$$\psi_k(\boldsymbol{x},\boldsymbol{\theta}) = \begin{cases} \frac{\partial \ln f(\boldsymbol{x},\boldsymbol{\theta})}{\partial \lambda_k}, & \text{if } \boldsymbol{\theta}\in\Theta' \\ 0, & \text{if } \boldsymbol{\theta}\notin\Theta' \end{cases} \tag{9}$$

where $\Theta' = \{\boldsymbol{\theta}\in\Theta : f(\boldsymbol{x},\boldsymbol{\theta})>0 \text{ a.e. } \boldsymbol{x}\in\Omega\}$. Conversely, for k restricted to $\{1,\ldots,K\}$ (changepoint locations), we define these measurable functions as for the WW bound, i.e.,

$$\psi_{K+1+k}(\boldsymbol{x},\boldsymbol{\theta}) = \sqrt{\frac{f(\boldsymbol{x},\boldsymbol{\theta}+\boldsymbol{h}_k)}{f(\boldsymbol{x},\boldsymbol{\theta})}} - \sqrt{\frac{f(\boldsymbol{x},\boldsymbol{\theta}-\boldsymbol{h}_k)}{f(\boldsymbol{x},\boldsymbol{\theta})}} \tag{10}$$

where $\boldsymbol{h}_k$ is any vector of size $2K+1$ of the form $\boldsymbol{h}_k = \left[\boldsymbol{0}_{K+1}^T,\, \boldsymbol{0}_{k-1}^T, h_k, \boldsymbol{0}_{K-k}^T\right]^T$, for $k=1,\ldots,K$, in which $\boldsymbol{0}_k$ denotes the zero vector of length k. Note that the value of h_k can be arbitrarily chosen by the user as far as it allows the invertibility of $\boldsymbol{P}$.

The next step in our analysis is to derive the matrix V. Denote as $\boldsymbol{V}_{22}$ the $K\times K$ diagonal matrix whose elements are, for any $k\in\{1,\ldots,K\}$

$$[\boldsymbol{V}_{22}]_{k,k} = -h_k \mathbb{E}_{\boldsymbol{x},\boldsymbol{\theta}}\left\{\sqrt{\frac{f(\boldsymbol{x},\boldsymbol{\theta}+\boldsymbol{h}_k)}{f(\boldsymbol{x},\boldsymbol{\theta})}}\right\}. \tag{11}$$

Substituting (9) and (10) into (7), we obtain

$$\boldsymbol{V} = \begin{bmatrix} -\boldsymbol{I}_{K+1} & \boldsymbol{0}_{(K+1)\times K} \\ \boldsymbol{0}_{K\times(K+1)} & \boldsymbol{V}_{22} \end{bmatrix} \tag{12}$$

where $\boldsymbol{I}_{K+1}$ denotes the $(K+1)\times(K+1)$ identity matrix and $\boldsymbol{0}_{(K+1)\times K}$ is the $(K+1)\times K$ zero matrix, provided the following conditions are satisfied

1) $f(\boldsymbol{x},\boldsymbol{\theta})$ is absolutely continuous w.r.t. λ_k, $k=1,\ldots,K+1$, a.e. $x\in\Omega$;
2) $\lim_{\lambda_k\to 0}\lambda_k f(\boldsymbol{x},\boldsymbol{\theta}) = \lim_{\lambda_k\to+\infty}\lambda_k f(\boldsymbol{x},\boldsymbol{\theta}) = 0$, $k=1,\ldots,K+1$, a.e. $x\in\Omega$.

Note that these two conditions correspond to the necessary and usual regularity conditions for the derivation of the Bayesian CR bound.

Similarly, by plugging (9) and (10) into (8), we obtain the expression of the matrix $\boldsymbol{P}$, which can be split into four blocks as follows

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{12}^T & \boldsymbol{P}_{22} \end{bmatrix} \tag{13}$$

in which $\boldsymbol{P}_{11}$ is the $(K+1)\times(K+1)$ matrix whose elements are

$$[\boldsymbol{P}_{11}]_{k,l} = \mathbb{E}_{\boldsymbol{x},\boldsymbol{\theta}}\left\{\frac{\partial \ln f(\boldsymbol{x},\boldsymbol{\theta})}{\partial\lambda_k}\frac{\partial \ln f(\boldsymbol{x},\boldsymbol{\theta})}{\partial\lambda_l}\right\} \tag{14}$$

$\boldsymbol{P}_{12}$ is the $(K+1)\times K$ matrix whose elements are

$$[\boldsymbol{P}_{12}]_{k,l} = \mathbb{E}_{\boldsymbol{x},\boldsymbol{\theta}}\left\{\frac{\partial \ln f(\boldsymbol{x},\boldsymbol{\theta})}{\partial\lambda_k}\left(\sqrt{\frac{f(\boldsymbol{x},\boldsymbol{\theta}+\boldsymbol{h}_l)}{f(\boldsymbol{x},\boldsymbol{\theta})}} - \sqrt{\frac{f(\boldsymbol{x},\boldsymbol{\theta}-\boldsymbol{h}_l)}{f(\boldsymbol{x},\boldsymbol{\theta})}}\right)\right\} \tag{15}$$

and finally $\boldsymbol{P}_{22}$ is the $K\times K$ matrix whose elements are

$$[\boldsymbol{P}_{22}]_{k,l} = \mathbb{E}_{\boldsymbol{x},\boldsymbol{\theta}}\left\{\left(\sqrt{\frac{f(\boldsymbol{x},\boldsymbol{\theta}+\boldsymbol{h}_k)}{f(\boldsymbol{x},\boldsymbol{\theta})}} - \sqrt{\frac{f(\boldsymbol{x},\boldsymbol{\theta}-\boldsymbol{h}_k)}{f(\boldsymbol{x},\boldsymbol{\theta})}}\right)\left(\sqrt{\frac{f(\boldsymbol{x},\boldsymbol{\theta}+\boldsymbol{h}_l)}{f(\boldsymbol{x},\boldsymbol{\theta})}} - \sqrt{\frac{f(\boldsymbol{x},\boldsymbol{\theta}-\boldsymbol{h}_l)}{f(\boldsymbol{x},\boldsymbol{\theta})}}\right)\right\}. \tag{16}$$

Note that the same bound can be obtained by defining all the functions ψ_k for $k\in\{1,\ldots,2K+1\}$ as in (10), with $\boldsymbol{h}_k = \left[\boldsymbol{0}_{k-1}^T, h_k, \boldsymbol{0}_{2K+1-k}^T\right]^T$, by letting h_k tend to 0 for $k=1,\ldots,K+1$, and by using a Taylor expansion. Finally, the tightest lower bound is obtained by maximizing the right hand side of (6) w.r.t. $h_1,\ldots,h_K$.

IV. Application to multiple change-points in Poisson Time-Series

This section presents the main results about the derivation of the lower bound presented in Section III for the problem formulated in Section II. The full calculation details are provided in the appendices. The joint distribution of the observation and parameter vectors can be expressed as

$$f\left(\boldsymbol{x}=\boldsymbol{\kappa},\boldsymbol{\theta}\right)=f\left(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}\right)f\left(\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}\right). \tag{17}$$

After plugging (2), (3) and (4) into (17), we can deduce the expressions of $f\left(\boldsymbol{x},\boldsymbol{\theta}+\boldsymbol{h}_k\right)$ for $k=1,\ldots,K$, and $\partial f\left(\boldsymbol{x},\boldsymbol{\theta}\right)/\partial\lambda_k$ for $k=1,\ldots,K+1$. Let us first introduce some useful notations for the following mathematical functions. We first define the function $\varphi_{h_k}(\boldsymbol{y})$ of the vector $\boldsymbol{y}=[y_1,y_2]^T\in\mathbb{R}_+^2$ as

$$\varphi_{h_k}\left(\boldsymbol{y}\right)=y_1^{\alpha_k-1}y_2^{\alpha_{k+1}-1}\exp\left\{-\beta\left(y_1+y_2\right)-|h_k|\frac{\left(\sqrt{y_2}-\sqrt{y_1}\right)^2}{2}\right\} \tag{18}$$

and the following integral as

$$\Phi\left(h_k\right)=\frac{\beta^{\alpha_k+\alpha_{k+1}}}{\Gamma\left(\alpha_k\right)\Gamma\left(\alpha_{k+1}\right)}\int_{\mathbb{R}_+^2}\varphi_{h_k}\left(\boldsymbol{y}\right)\mathrm{d}\boldsymbol{y}. \tag{19}$$

We also define the function $\phi_{h_k,h_{k+1}}(\boldsymbol{z})$ of the vector $\boldsymbol{z}=[z_1,z_2,z_3]^T\in\mathbb{R}_+^3$ as the trivariate version of φ_{h_k}, i.e.,

$$\phi_{h_k,h_{k+1}}\left(\boldsymbol{z}\right)=z_1^{\alpha_k-1}z_2^{\alpha_{k+1}-1}z_3^{\alpha_{k+2}-1}\exp\left\{-\beta\left(z_1+z_2+z_3\right)-|h_k|\frac{\left(\sqrt{z_2}-\sqrt{z_1}\right)^2}{2}-|h_{k+1}|\frac{\left(\sqrt{z_3}-\sqrt{z_2}\right)^2}{2}\right\}. \tag{20}$$

We finally define the three functions u, v and w as follows

$$u\left(\tau,h_k\right)=\begin{cases}\frac{(\tau-|h_k|)^2}{\tau^2} & \text{if } k\leq K-1 \text{ and } |h_k|\leq\tau\\ \frac{\tau-|h_K|}{\tau} & \text{if } k=K \text{ and } |h_K|\leq\tau\\ 0 & \text{if } |h_k|>\tau\end{cases} \tag{21}$$

$$v\left(\tau,h_k,h_{k+1}\right)=\begin{cases}\frac{(\tau-|h_k|)(\tau-|h_{k+1}|)}{\tau^3} & \text{if } k\leq K-1\\ & \text{and } \max\left(|h_k|,|h_{k+1}|\right)\leq\tau;\\ \frac{\tau-|h_K|}{\tau^2} & \text{if } k=K \text{ and } |h_K|\leq\tau;\\ 0 & \text{if } \max\left(|h_k|,|h_{k+1}|\right)>\tau,\end{cases} \tag{22}$$

$$\begin{aligned}w\left(\boldsymbol{z},\tau,h_k,h_{k+1}\right)=&\,2\max\left(\tau-|h_k|-|h_{k+1}|,0\right)-\max\bigl(\tau-\max\left(|h_k|,|h_{k+1}|\right),0\bigr)\\&-\max\left(\tau-|h_k|-|h_{k+1}|+1,0\right)+\frac{1-r^{1-\min(|h_k|,|h_{k+1}|)}\left(\boldsymbol{z}\right)}{1-r\left(\boldsymbol{z}\right)}\end{aligned} \tag{23}$$

in which $r\left(\boldsymbol{z}\right)=\exp\left\{-z_2+\sqrt{z_1z_2}+\sqrt{z_2z_3}-\sqrt{z_1z_3}\right\}$. Using these functions, we now give the expressions of the matrix blocks composing $\boldsymbol{V}$ and $\boldsymbol{P}$, i.e., $\boldsymbol{V}_{22}$, $\boldsymbol{P}_{11}$, $\boldsymbol{P}_{12}$, $\boldsymbol{P}_{21}$ and $\boldsymbol{P}_{22}$ which were introduced in Section III.

After plugging (17) into (11) and computing the expectations, we obtain, for $k=1,\ldots,K$

$$[\boldsymbol{V}_{22}]_{k,k}=-h_k\,u\left(\tau,h_k\right)\Phi\left(h_k\right). \tag{24}$$

The expression of $\boldsymbol{P}_{11}$ is obtained by substituting (17) into (14), which leads to a diagonal matrix whose elements have the following form (for $\alpha_k>2$)

$$[\boldsymbol{P}_{11}]_{k,k}=\left(\frac{\beta\left(\tau+1\right)}{2\left(\alpha_k-1\right)}+\frac{\beta^2}{\alpha_k-2}\right). \tag{25}$$

Similarly, the expressions of $\boldsymbol{P}_{12}$ (of size $(K+1)\times K$) and $\boldsymbol{P}_{21}$ (of size $K\times(K+1)$) can be obtained after plugging (17) into (15), which leads to

$$\boldsymbol{P}_{12}=\boldsymbol{P}_{21}^T=\begin{bmatrix}A_{1,1} & 0 & \cdots & 0\\ A_{2,1} & A_{2,2} & \ddots & \vdots\\ 0 & A_{3,2} & \ddots & 0\\ \vdots & \ddots & \ddots & A_{K,K}\\ 0 & \cdots & 0 & A_{K+1,K}\end{bmatrix} \tag{26}$$

where, for $k=1,\ldots,K$, and $l=k$ or $l=k-1$,

$$A_{k,l}=\pm h_l\,u\left(\tau,h_l\right)\frac{\beta^{\alpha_l+\alpha_{l+1}}}{\Gamma(\alpha_l)\Gamma(\alpha_{l+1})}\int_{\mathbb{R}_+^2}\left(\sqrt{\left(\frac{y_1}{y_2}\right)^{\pm1}}-1\right)\varphi_{h_l}\left(\boldsymbol{y}\right)\mathrm{d}\boldsymbol{y} \tag{27}$$

where both "±" signs are "+" signs if $l = k-1$, and they are "−" signs if $l = k$.

Finally, by substituting (17) into (16), the matrix $\boldsymbol{P}_{22}$ can be written as the following symmetric tridiagonal matrix

$$\boldsymbol{P}_{22} = \begin{bmatrix} B_1 & C_1 & 0 & \cdots & 0 \\ C_1 & B_2 & C_2 & \ddots & \vdots \\ 0 & C_2 & B_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & C_{K-1} \\ 0 & \cdots & 0 & C_{K-1} & B_K \end{bmatrix} \tag{28}$$

where, for $k = 1, \ldots, K$,

$$B_k = 2\big(u(\tau, h_k) - u(\tau, 2h_k)\,\Phi(2h_k)\big) \tag{29}$$

and, for $k = 1, \ldots, K-1$,

$$C_k = v(\tau, h_k, h_{k+1}) \frac{\beta^{\alpha_k+\alpha_{k+1}+\alpha_{k+2}}}{\Gamma(\alpha_k)\,\Gamma(\alpha_{k+1})\,\Gamma(\alpha_{k+2})} \int_{\mathbb{R}_+^3} \phi_{h_k,h_{k+1}}(\boldsymbol{z})\, w(\boldsymbol{z}, \tau, h_k, h_{k+1})\mathrm{d}\boldsymbol{z}. \tag{30}$$

A. Practical computation of the bound

The lower bound given by the right-hand side of (6), that we will denote by $\boldsymbol{R}$ can be computed using the previous formulas, i.e., from (24) to (30). It can be noticed that some integrals (in (24), (27) and (30)), do not have any closed-form expression requiring some numerical scheme for their computation. In this paper, we have used the adaptive quadrature method [15] that proved efficient for our computations.

In addition, we would like to stress that, even if it does not appear explicitly with the adopted notations, the matrix $\boldsymbol{R}$ actually depends on the parameters $\alpha_1, \ldots, \alpha_{K+1}, \beta, \tau, h_1, \ldots, h_K$ (only the dependency on $h_1, \ldots, h_K$ has been mentioned from (18) to (23)). Since each vector $\boldsymbol{h} = (h_1, \ldots, h_K)$ leads to a lower bound $\boldsymbol{R}(\boldsymbol{h})$, one obtains a finite set of lower bounds $\mathcal{W} = \{\boldsymbol{R}(\boldsymbol{h}) \mid \boldsymbol{h} \in \mathcal{H}\}$, in which $\mathcal{H}$ is the set of all possible values of $\boldsymbol{h}$. As already mentioned, the proper Cramér-Rao/Weiss-Weinstein lower bound is the tightest value of $\boldsymbol{R}(\boldsymbol{h})$, namely the supremum of $\mathcal{W}$, that we denote by $\boldsymbol{B} = \sup(\mathcal{W}) = \sup_{h_1,\ldots,h_K} \boldsymbol{R}(\boldsymbol{h})$. The supremum operation has to be taken w.r.t. the Loewner partial ordering, denoted by "$\preceq$" [16]. This ordering implies that a unique supremum in the finite set $\mathcal{W}$ might not exist. However, it is possible to approximate this supremum by computing a minimal upper-bound $\boldsymbol{B}^*$ of the set $\mathcal{W}$: this bound is such that, for all $\boldsymbol{h} \in \mathcal{H}$, $\boldsymbol{B}^* \succeq \boldsymbol{R}(\boldsymbol{h})$, and there is no smaller matrix $\boldsymbol{B}' \preceq \boldsymbol{B}^*$ that also verifies $\boldsymbol{B}' \succeq \boldsymbol{R}(\boldsymbol{h})$, $\forall \boldsymbol{h} \in \mathcal{H}$. It has been shown in [10], [17] that finding $\boldsymbol{B}^*$ is equivalent to finding the minimum volume hyper-ellipsoid $\varepsilon(\boldsymbol{B}^*) = \{\boldsymbol{x} \in \mathbb{R}^K \mid \boldsymbol{x}^T \boldsymbol{B}^* \boldsymbol{x} \leq 1\}$ that covers the union of hyper-ellipsoids $\varepsilon\big(\boldsymbol{R}(\boldsymbol{h})\big) = \{\boldsymbol{x} \in \mathbb{R}^K \mid \boldsymbol{x}^T \boldsymbol{R}(\boldsymbol{h}) \boldsymbol{x} \leq 1\}$. The search of this ellipsoid can actually be formulated as the following convex optimization problem [17]:

$$\begin{array}{ll} \text{minimize} & \log\Big(\det\big(\boldsymbol{B}^{1/2}\big)\Big) \\ \text{subject to} & \begin{cases} b_1 \geq 0, b_2 \geq 0, \ldots, b_{N_{\boldsymbol{h}}} \geq 0, \\ \begin{bmatrix} \boldsymbol{B}^{-1} - b_n\big(\boldsymbol{R}(\boldsymbol{h})_n\big)^{-1} & \boldsymbol{0}_{(2K+1)\times 1} \\ \boldsymbol{0}_{1\times(2K+1)} & b_n - 1 \end{bmatrix} \preceq \boldsymbol{0}_{2K+2} \\ \qquad\qquad\qquad\qquad (n = 1, \ldots, N_{\boldsymbol{h}}) \end{cases} \end{array} \tag{31}$$

in which $N_{\boldsymbol{h}}$ denotes the number of elements of the set $\mathcal{H}$, and $\boldsymbol{R}(\boldsymbol{h})_n \in \mathcal{W}$ is an indexed version of $\boldsymbol{R}(\boldsymbol{h})$ (i.e., when n varies from 1 to $N_{\boldsymbol{h}}$, $\boldsymbol{h}$ runs through all the possible combinations of $h_1, \ldots, h_K$, and $\boldsymbol{R}(\boldsymbol{h})_n$ runs through all the elements of $\mathcal{W}$). The problem (31) can be solved efficiently using a semidefinite programming tool, such as the one provided in the CVX package [18].

V. Numerical results

This section analyzes the evolution of the proposed bound as a function of a parameter that is classically used for changepoint estimation performance. This parameter is either referred to as "amount of change" [19], "magnitude of change" or "signal-to-noise ratio" (SNR): in [9]–[11], for Poisson distributed data, the SNR is defined as $\nu = (\lambda_{k+1} - \lambda_k)^2/\lambda_k^2$, for $k = 1, \ldots, K+1$. In our context, since each λ_k is a random variable with a gamma distribution of parameters α_k and β (as stated in (2)), this leads to a lower bound $\boldsymbol{B}$ that does not depend on $\lambda_1, \ldots, \lambda_{K+1}$, and *a fortiori* on ν. However, the bound depends upon the parameters α_k and β, which can then be used to drive the average $(\lambda_k)_{\mathsf{mean}} = \alpha_k/\beta$ generated by the Gamma prior. Thus, by substituting $(\lambda_k)_{\mathsf{mean}}$ with λ_k in the definition of ν, we obtain $\bar{\nu} = (\alpha_{k+1} - \alpha_k)^2/\alpha_k^2$. Such a definition implies that the higher $\bar{\nu}$, the higher the amount of change between two consecutive segments, *on average*.

In this study, we present some simulation results obtained for $T = 80$ observations, $K = 1$ change, and with $\bar{\nu}$ ranging from -20 dB to 15 dB. Such a choice for the value of K is justified by the fact that it yields less complex expressions of

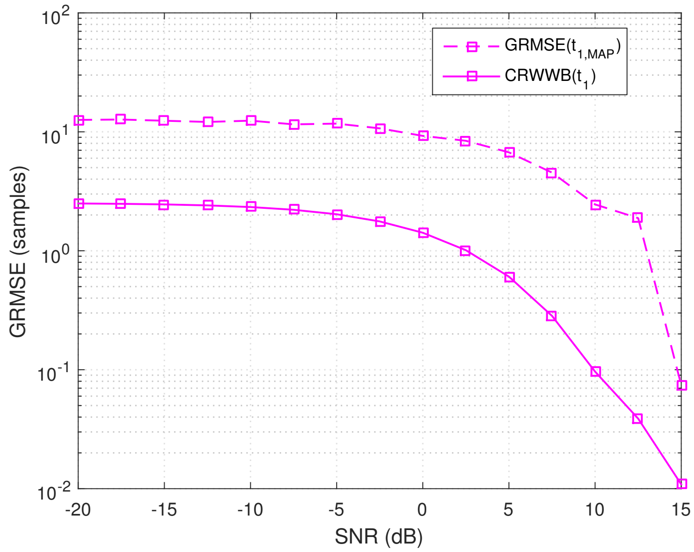

Fig. 1. Estimated GRMSE and proposed lower bound w.r.t. the change-point t_1, versus SNR, with $T = 80$ snapshots and $K = 1$ change in the mean rate of a Poisson time series.

the estimators given in (32) and (33). We chose $\alpha_1 = 3$, and the subsequent α_2 is given by $\alpha_2 = \alpha_1(1 + \sqrt{\bar{\nu}})$. We compare the proposed bound with the estimated global mean square error (GMSE) of the maximum a posteriori (MAP) estimator of $\boldsymbol{\theta} = [\lambda_1, \lambda_2, t_1]^T$. It is worth mentioning that, given the posterior density $f(\boldsymbol{\lambda}, \boldsymbol{t}|\boldsymbol{x} = \boldsymbol{\kappa})$ (that is proportional to (17)), there is a closed-form expression of the MAP estimator of $\boldsymbol{\lambda}$, for a given $\boldsymbol{\ell}$, that is, for $k = 1, \ldots, K+1$:

$$\hat{\lambda}_k^{\mathsf{MAP}}(\ell_{k-1}, \ell_k) = \frac{\alpha_k + \left(\sum_{t=\ell_{k-1}+1}^{\ell_k} \kappa_t\right) - 1}{\beta + (\ell_k - \ell_{k-1})}. \tag{32}$$

This closed form expression is then used to obtain the MAP estimator of $\boldsymbol{t}$

$$\hat{\boldsymbol{t}}^{\mathsf{MAP}} = \arg\max_{\boldsymbol{\ell}} \ln f\left(\hat{\boldsymbol{\lambda}}^{\mathsf{MAP}}(\boldsymbol{\ell}), \boldsymbol{t} = \boldsymbol{\ell} | \boldsymbol{x} = \boldsymbol{\kappa}\right). \tag{33}$$

The estimated global root mean square error (GRMSE) of $\hat{\boldsymbol{t}}^{\mathsf{MAP}}$ computed using 1000 Monte-Carlo runs and the associated lower bound are compared in Fig. 1. Even if there exists a gap between the GRMSE and the bound, the difference decreases as $\bar{\nu}$ increases: at $\bar{\nu} = 10$ dB, the difference in terms of number of samples is no more than 3 samples; at $\bar{\nu} = 15$ dB, it is less than 0.1 samples. The MAP behavior even seems to be closer to the bound for $\bar{\nu} \geq 15$ dB. However, it could not be displayed for numerical reasons, the GRMSE tending steeply to zero. Finally, the derived bound provides a fair approximation of the changepoint estimation behavior, in this context of Poisson data when the Poisson parameters λ_k are unknown.

Appendix
Details about the derivation of the bound

In this section, we give all the calculation details leading to the bound given in Section IV.

A. Derivation of $\boldsymbol{V}_{22}$ and $\boldsymbol{P}_{22}$

Let us first remark that

$$[\boldsymbol{V}_{22}]_{k,k} = -h_k \zeta(\boldsymbol{h}_k, \boldsymbol{0}_{2K+1}) \qquad \text{and} \qquad [\boldsymbol{P}_{22}]_{k,l} = \zeta(\boldsymbol{h}_k, \boldsymbol{h}_l) + \zeta(-\boldsymbol{h}_k, -\boldsymbol{h}_l) - \zeta(-\boldsymbol{h}_k, \boldsymbol{h}_l) - \zeta(\boldsymbol{h}_k, -\boldsymbol{h}_l) \tag{34}$$

in which $\zeta\left(\boldsymbol{h}_k, \boldsymbol{h}'_l\right)$, for $k \in \{1, \ldots, K\}$, denotes

$$\begin{aligned}\zeta\left(\boldsymbol{h}_k, \boldsymbol{h}'_l\right) &= \mathbb{E}_{\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{t}}\left\{\frac{\sqrt{f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_k) f\left(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}'_l\right)}}{f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})}\right\} \\ &= \sum_{\boldsymbol{\ell} \in \mathbb{Z}^K} \int_{\mathbb{R}_+^{K+1}} \sum_{\boldsymbol{\kappa} \in \mathbb{N}^T} \sqrt{f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_k) f\left(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}'_l\right)} \mathrm{d}\boldsymbol{\lambda}\end{aligned} \tag{35}$$

with vectors $\boldsymbol{h}_k$ and $\boldsymbol{h}'_l$ of the form $\boldsymbol{h}_k = \left[\mathbf{0}_{k-1}^T, h_k, \mathbf{0}_{K-k}^T\right]^T$ and $\boldsymbol{h}'_l = \left[\mathbf{0}_{l-1}^T, h'_l, \mathbf{0}_{K-l}^T\right]^T$, both with size K. Note that function ζ, even though it is written as a function of two vectors $\boldsymbol{h}_k$ and $\boldsymbol{h}'_l$, is actually a function of the two scalars h_k and h'_l (which are the non zero components of the two aforementioned vectors). Throughout the following developments, we will either use $\zeta(\boldsymbol{h}_k, \boldsymbol{h}'_l)$ or $\zeta(h_k, h'_l)$, depending on the convenience.

Since $\boldsymbol{V}_{22}$ is diagonal and $\boldsymbol{P}_{22}$ is symmetric, we can assume $l \geq k$, without loss of generality.

Developing each probability density function (p.d.f.) in (35) from (17), we have

$$\begin{aligned}\zeta(\boldsymbol{h}_k, \boldsymbol{h}'_l) &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^K} \int_{\mathbb{R}_+^{K+1}} \left[\sqrt{f(\boldsymbol{\lambda}, \boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_k) f(\boldsymbol{\lambda}, \boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}'_l)} \sum_{\boldsymbol{\kappa}\in\mathbb{N}^T} \sqrt{f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda}, \boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_k) f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda}, \boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}'_l)}\right] \mathrm{d}\boldsymbol{\lambda} \\ &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^K} \int_{\mathbb{R}_+^{K+1}} \pi(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_l)\, \acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_l)\, \mathrm{d}\boldsymbol{\lambda} \end{aligned} \tag{36}$$

in which

$$\pi(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_l) = \sqrt{f(\boldsymbol{\lambda}, \boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_k) f(\boldsymbol{\lambda}, \boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}'_l)} \tag{37}$$

and

$$\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_l) = \sum_{\boldsymbol{\kappa}\in\mathbb{N}^T} \sqrt{f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda}, \boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_k) f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda}, \boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}'_l)}. \tag{38}$$

We first calculate (37), and then (38).

1) Derivation of $\pi(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_l)$*:* From (2) and (3), one can deduce

$$\begin{aligned}\pi(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_l) = &\prod_{i=1}^{K+1} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} \exp(-\beta\lambda_i)\, \mathbb{I}_{\mathbb{R}_+}(\lambda_i) \\ &\times \frac{1}{\tau^K} \left(\prod_{i=1}^{K} \mathbb{I}_{\{\ell_{i-1}+1,\ldots,\ell_{i-1}+\tau\}}(\ell_i)\right) \mathbb{I}_{\{\ell_{k-1}-h_k+1,\ldots,\ell_{k-1}-h_k+\tau\}}(\ell_k)\, \mathbb{I}_{\{\ell_k+h_k+1,\ldots,\ell_k+h_k+\tau\}}(\ell_{k+1}) \\ &\times \left(\prod_{i=1}^{K} \mathbb{I}_{\{\ell_{i-1}+1,\ldots,\ell_{i-1}+\tau\}}(\ell_i)\right) \mathbb{I}_{\left\{\ell_{l-1}-h'_l+1,\ldots,\ell_{l-1}-h'_l+\tau\right\}}(\ell_l)\, \mathbb{I}_{\left\{\ell_l+h'_l+1,\ldots,\ell_l+h'_l+\tau\right\}}(\ell_{l+1}) \\ = &\prod_{i=1}^{K+1} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} \exp(-\beta\lambda_i)\, \mathbb{I}_{\mathbb{R}_+}(\lambda_i) \cdot \frac{1}{\tau^K} \mathbb{I}_{\mathcal{J}_{h_k,h'_l}}(\boldsymbol{\ell}) \end{aligned} \tag{39}$$

in which $\mathcal{J}_{h_k,h'_l}$, for $l > k+1$, denotes the following set (that is a subset of $\mathbb{N}^K$)

$$\begin{aligned}\mathcal{J}_{h_k,h'_l} = \left(\prod_{\substack{i=1\\ i\neq k,k+1\\ i\neq l,l+1}}^{K} \{\ell_{i-1}+1,\ldots,\ell_{i-1}+\tau\}\right) &\times \left(\{\ell_{k-1}+1,\ldots,\ell_{k-1}+\tau\} \cap \{\ell_{k-1}-h_k+1,\ldots,\ell_{k-1}-h_k+\tau\}\right) \\ &\times \left(\{\ell_k+1,\ldots,\ell_k+\tau\} \cap \{\ell_k+h_k+1,\ldots,\ell_k+h_k+\tau\}\right) \\ &\times \left(\{\ell_{l-1}+1,\ldots,\ell_{l-1}+\tau\} \cap \{\ell_{l-1}-h'_l+1,\ldots,\ell_{l-1}-h'_l+\tau\}\right) \\ &\times \left(\{\ell_l+1,\ldots,\ell_l+\tau\} \cap \{\ell_l+h'_l+1,\ldots,\ell_l+h'_l+\tau\}\right), \end{aligned} \tag{40}$$

for $l = k+1$, it is

$$\begin{aligned}\mathcal{J}_{h_k,h'_{k+1}} = \left(\prod_{\substack{i=1\\ i\neq k,k+1,k+2}}^{K} \{\ell_{i-1}+1,\ldots,\ell_{i-1}+\tau\}\right) &\times \left(\{\ell_{k-1}+1,\ldots,\ell_{k-1}+\tau\} \cap \{\ell_{k-1}-h_k+1,\ldots,\ell_{k-1}-h_k+\tau\}\right) \\ &\times \left(\{\ell_k+1,\ldots,\ell_k+\tau\} \cap \{\ell_k+h_k+1,\ldots,\ell_k+h_k+\tau\} \right.\\ &\qquad \left. \cap \left\{\ell_k-h'_{k+1}+1,\ldots,\ell_k-h'_{k+1}+\tau\right\}\right) \\ &\times \left(\{\ell_{k+1}+1,\ldots,\ell_{k+1}+\tau\} \right.\\ &\qquad \left. \cap \left\{\ell_{k+1}+h'_{k+1}+1,\ldots,\ell_{k+1}+h'_{k+1}+\tau\right\}\right), \end{aligned} \tag{41}$$

and for $l = k$, it is

$$\begin{aligned}\mathcal{J}_{h_k,h'_k} = \left(\prod_{\substack{i=1\\ i\neq k,k+1}}^{K} \{\ell_{i-1}+1,\ldots,\ell_{i-1}+\tau\}\right) &\times \left(\{\ell_{k-1}+1,\ldots,\ell_{k-1}+\tau\} \cap \{\ell_{k-1}-h_k+1,\ldots,\ell_{k-1}-h_k+\tau\} \right.\\ &\qquad \left. \cap \left\{\ell_{k-1}-h'_k+1,\ldots,\ell_{k-1}-h'_k+\tau\right\}\right) \\ &\times \left(\{\ell_k+1,\ldots,\ell_k+\tau\} \cap \{\ell_k+h_k+1,\ldots,\ell_k+h_k+\tau\} \right.\\ &\qquad \left. \cap \left\{\ell_k+h'_k+1,\ldots,\ell_k+h'_k+\tau\right\}\right), \end{aligned} \tag{42}$$

with $\ell_0 = 0$.

TABLE I
LOWEST ELEMENTS $(\ell_i)_{\min}$ AND GREATEST ELEMENTS $(\ell_i)_{\max}$ OF THE SETS $\mathcal{I}_i$ FOR $i = 1, \ldots, K$, IN THE CASE UT $(l > k+1)$.

i	$(\ell_i)_{\min}$	$(\ell_i)_{\max}$
k	$\ell_{k-1} + 1 + \max(-h_k, 0)$	$\ell_{k-1} + \tau - \max(h_k, 0)$
$k+1$	$\ell_k + 1 + \max(h_k, 0)$	$\ell_k + \tau - \max(-h_k, 0)$
l	$\ell_{l-1} + 1 + \max(-h'_l, 0)$	$\ell_{l-1} + \tau - \max(h'_l, 0)$
$l+1$ (if $l+1 \leq K$)	$\ell_l + 1 + \max(h'_l, 0)$	$\ell_l + \tau - \max(-h'_l, 0)$
$i \neq k, k+1, l, l+1$	$\ell_{i-1} + 1$	$\ell_{i-1} + \tau$

TABLE II
LOWEST ELEMENTS $(\ell_i)_{\min}$ AND GREATEST ELEMENTS $(\ell_i)_{\max}$ OF THE SETS $\mathcal{I}_i$ FOR $i = 1, \ldots, K$, IN THE CASE FSD $(l = k+1)$.

i	$(\ell_i)_{\min}$	$(\ell_i)_{\max}$
k	$\ell_{k-1} + 1 + \max(-h_k, 0)$	$\ell_{k-1} + \tau - \max(h_k, 0)$
$k+1$	$\ell_k + 1 + \max(h_k, -h'_{k+1}, 0)$	$\ell_k + \tau - \max(-h_k, h'_{k+1}, 0)$
$k+2$ (if $k+2 \leq K$)	$\ell_{k+1} + 1 + \max(h'_{k+1}, 0)$	$\ell_{k+1} + \tau - \max(-h'_{k+1}, 0)$
$i \neq k, k+1, k+2$	$k_{i-1} + 1$	$k_{i-1} + \tau$

TABLE III
LOWEST ELEMENTS $(\ell_i)_{\min}$ AND GREATEST ELEMENTS $(\ell_i)_{\max}$ OF THE SETS $\mathcal{I}_i$ FOR $i = 1, \ldots, K$, IN THE CASE D $(l = k)$.

i	$(\ell_i)_{\min}$	$(\ell_i)_{\max}$
k	$\ell_{k-1} + 1 + \max(-h_k, -h'_k, 0)$	$\ell_{k-1} + \tau - \max(h_k, h'_k, 0)$
$k+1$ (if $k+1 \leq K$)	$\ell_k + 1 + \max(h_k, h'_k, 0)$	$\ell_k + \tau - \max(-h_k, -h'_k, 0)$
$i \neq k, k+1$	$\ell_{i-1} + 1$	$\ell_{i-1} + \tau$

To summarize this, and in an effort to make it more explicit, the set $\mathcal{J}_{h_k,h'_l} \subset \mathbb{N}^K$ can be written as the cartesian product of K sets $\mathcal{I}_i \subset \mathbb{N}$ of consecutive integers, with $i = 1, \ldots K$:

$$\mathcal{J}_{h_k,h'_l} = \prod_{i=1}^{K} \mathcal{I}_i. \tag{43}$$

The lowest elements $(\ell_i)_{\min} = \min \mathcal{I}_i$ and the greatest elements $(\ell_i)_{\max} = \max \mathcal{I}_i$ of the sets $\mathcal{I}_i$ for $i = 1, \ldots, K$ are given in tables I, II, and III for the three cases i) $l > k+1$ (case "UT", for "upper triangle"), ii) $l = k+1$ (case "FSD", for "first subdiagonal"), and iii) $l = k$ (case "D", for "diagonal"), respectively.

2) Derivation of $\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_l)$: It follows directly from (4) that

$$f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_k) = \prod_{\substack{i=1 \\ i \neq k, k+1}}^{K+1} \left[\prod_{t=\ell_{i-1}+1}^{\ell_i} \sqrt{\frac{\lambda_i^{\kappa_t}}{\kappa_t!} \exp(-\lambda_i)} \right] \prod_{t=\ell_{k-1}+1}^{\ell_k + h_k} \sqrt{\frac{\lambda_k^{\kappa_t}}{\kappa_t!} \exp(-\lambda_k)} \prod_{t=\ell_k + h_k + 1}^{\ell_{k+1}} \sqrt{\frac{\lambda_{k+1}^{\kappa_t}}{\kappa_t!} \exp(-\lambda_{k+1})} \tag{44}$$

and

$$f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}'_l) = \prod_{\substack{i=1 \\ i \neq l, l+1}}^{K+1} \left[\prod_{t=\ell_{i-1}+1}^{\ell_i} \sqrt{\frac{\lambda_i^{\kappa_t}}{\kappa_t!} \exp(-\lambda_i)} \right] \prod_{t=\ell_{l-1}+1}^{\ell_l + h'_l} \sqrt{\frac{\lambda_l^{\kappa_t}}{\kappa_t!} \exp(-\lambda_l)} \prod_{t=\ell_l + h'_l + 1}^{\ell_{l+1}} \sqrt{\frac{\lambda_{l+1}^{\kappa_t}}{\kappa_t!} \exp(-\lambda_{l+1})} \tag{45}$$

with $\ell_0 = 0$ and $\ell_{K+1} = T$. By plugging (44) and (45) into (38), one obtains the expression of $\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_l)$, whose writing actually depends on the three aforementioned cases, i.e., i) UT $(l > k+1)$, ii) FSD $(l = k+1)$, and iii) D $(l = k)$.

i) Case UT ($l > k+1$): In this case, by plugging (44) and (45) into (38), one obtains:

$$\begin{aligned}\acute{\zeta}(\boldsymbol{\lambda},\boldsymbol{\ell},\boldsymbol{h}_k,\boldsymbol{h}'_l) = \sum_{\kappa_1=0}^{+\infty}\cdots\sum_{\kappa_T=0}^{+\infty}\Bigg(&\prod_{\substack{i=1\\ i\neq k,k+1\\ i\neq l,l+1}}^{K+1}\left[\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]\prod_{t=\ell_{k-1}+1}^{\ell_k-\max(-h_k,0)}\frac{\lambda_k^{\kappa_t}}{\kappa_t!}\exp(-\lambda_k)\\
&\times\prod_{t=\ell_k-\max(-h_k,0)+1}^{\ell_k+\max(h_k,0)}\frac{\sqrt{\lambda_k\lambda_{k+1}}^{\kappa_t}}{\kappa_t!}\exp\left\{-\frac{\lambda_k+\lambda_{k+1}}{2}\right\}\prod_{t=\ell_k+\max(h_k,0)+1}^{\ell_{k+1}}\frac{\lambda_{k+1}^{\kappa_t}}{\kappa_t!}\exp(-\lambda_{k+1})\\
&\times\prod_{t=\ell_{l-1}+1}^{\ell_l-\max(-h'_l,0)}\frac{\lambda_l^{\kappa_t}}{\kappa_t!}\exp(-\lambda_l)\prod_{t=\ell_l-\max(-h'_l,0)+1}^{\ell_l+\max(h'_l,0)}\frac{\sqrt{\lambda_l\lambda_{l+1}}^{\kappa_t}}{\kappa_t!}\exp\left\{-\frac{\lambda_l+\lambda_{l+1}}{2}\right\}\\
&\times\prod_{t=\ell_l+\max(h'_l,0)+1}^{\ell_{l+1}}\frac{\lambda_{l+1}^{\kappa_t}}{\kappa_t!}\exp(-\lambda_{l+1})\Bigg)\end{aligned}\tag{46}$$

Note that the sums indices $\kappa_1,\kappa_2,\ldots,\kappa_{\ell_1},\kappa_{\ell_1+1},\ldots,\kappa_{\ell_2},\ldots,\kappa_{\ell_K},\ldots,\kappa_T$ are separated in (46). This implies that the sums of products become products of sums, and since

$$\sum_{\kappa_t=0}^{+\infty}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)=1,\tag{47}$$

then (46) becomes

$$\begin{aligned}\acute{\zeta}(\boldsymbol{\lambda},\boldsymbol{\ell},\boldsymbol{h}_k,\boldsymbol{h}'_l) &= \prod_{t=\ell_k-\max(-h_k,0)+1}^{\ell_k+\max(h_k,0)}\left(\exp\left\{-\frac{\lambda_k+\lambda_{k+1}}{2}\right\}\sum_{\kappa_t=0}^{+\infty}\frac{\sqrt{\lambda_k\lambda_{k+1}}^{\kappa_t}}{\kappa_t!}\right)\\
&\qquad\times\prod_{t=\ell_l-\max(-h'_l,0)+1}^{\ell_l+\max(h'_l,0)}\left(\exp\left\{-\frac{\lambda_l+\lambda_{l+1}}{2}\right\}\sum_{\kappa_t=0}^{+\infty}\frac{\sqrt{\lambda_l\lambda_{l+1}}^{\kappa_t}}{\kappa_t!}\right)\\
&= \exp\left\{-|h_k|\frac{\lambda_k+\lambda_{k+1}}{2}\right\}\left(\sum_{\kappa=0}^{+\infty}\frac{\sqrt{\lambda_k\lambda_{k+1}}^{\kappa}}{\kappa!}\right)^{|h_k|}\exp\left\{-|h'_l|\frac{\lambda_l+\lambda_{l+1}}{2}\right\}\left(\sum_{\kappa=0}^{+\infty}\frac{\sqrt{\lambda_l\lambda_{l+1}}^{\kappa}}{\kappa!}\right)^{|h'_l|}\\
&= \exp\left\{-|h_k|\frac{\lambda_k+\lambda_{k+1}}{2}\right\}\exp\left\{|h_k|\sqrt{\lambda_k\lambda_{k+1}}\right\}\exp\left\{-|h'_l|\frac{\lambda_l+\lambda_{l+1}}{2}\right\}\exp\left\{|h'_l|\sqrt{\lambda_l\lambda_{l+1}}\right\}\\
&= \exp\left\{-|h_k|\frac{\left(\sqrt{\lambda_{k+1}}-\sqrt{\lambda_k}\right)^2}{2}\right\}\exp\left\{-|h'_l|\frac{\left(\sqrt{\lambda_{l+1}}-\sqrt{\lambda_l}\right)^2}{2}\right\}\\
&= \rho^{|h_k|}(\lambda_k,\lambda_{k+1})\rho^{|h'_l|}(\lambda_l,\lambda_{l+1})\end{aligned}\tag{48}$$

in which, for $k=1,\ldots,K$,

$$\rho(\lambda_k,\lambda_{k+1})=\exp\left\{-\frac{\left(\sqrt{\lambda_{k+1}}-\sqrt{\lambda_k}\right)^2}{2}\right\}.\tag{49}$$

Note that $\acute{\zeta}(\boldsymbol{\lambda},\boldsymbol{\ell},\boldsymbol{h}_k,\boldsymbol{h}'_l)$ does actually not depend on $\boldsymbol{\ell}$ in this case.

ii) Case FSD ($l = k+1$): This case is more complicated than the two others, because the writing of (38) depends on whether $\ell_{k+1}+h'_{k+1}\gtrless\ell_k+h_k$ (note that in the case UT ($l>k+1$), one always has $\ell_l+h'_l>\ell_k+h_k$; this can be seen by analyzing table I). The case $\ell_{k+1}+h'_{k+1}<\ell_k+h_k$ is often referred to as "overlap case" in the following. It is first of interest to determine when this case occurs.

By analyzing the line "$i=k+1$" in table II, we can first remark that this case is possible only if $h_k>0$ and $h'_{k+1}<0$. In addition, since the set $\mathcal{I}_{k+1}$ depends on ℓ_k, the formal condition for the overlap case to occur can be written as:

$$\exists\ell_{k+1}\in\mathcal{I}_{k+1},\ \ell_{k+1}+h'_{k+1}<\ell_k+h_k,\tag{50}$$

which is equivalent to

$$(\ell_{k+1})_{\min}+h_{k+1}<\ell_k+h_k.\tag{51}$$

Using the expression of $(\ell_{k+1})_{\min}$ from table II, we obtain that the condition for the overlap case is finally

$$h_k > 0 \quad \text{and} \quad h'_{k+1} < 0 \quad \text{and} \quad \min(|h_k|, |h'_{k+1}|) \geq 2. \tag{52}$$

In the case whithout overlap, i.e., when the condition (52) is not met, the derivation of $\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_l)$ is exactly the same as in the case UT, which means we obtain

$$\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_{k+1}) = \rho^{|h_k|}(\lambda_k, \lambda_{k+1})\rho^{|h'_{k+1}|}(\lambda_{k+1}, \lambda_{k+2}). \tag{53}$$

On the other hand, when there is overlap, i.e., when the condition (52) is met, the derivation of (38) is done in the following way:

$$\begin{aligned}
\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_{k+1}) = \sum_{\kappa_1=0}^{+\infty} \cdots \sum_{\kappa_T=0}^{+\infty} \Bigg(& \prod_{\substack{i=1\\ i\neq k+1,k+2}}^{K+1} \left[\prod_{t=\ell_{i-1}+1}^{\ell_i} \frac{\lambda_i^{\kappa_t}}{\kappa_t!} \exp(-\lambda_i) \right] \prod_{t=\ell_k+1}^{\ell_{k+1}+h'_{k+1}} \frac{\sqrt{\lambda_k \lambda_{k+1}}^{\kappa_t}}{\kappa_t!} \exp\left\{ -\frac{\lambda_k + \lambda_{k+1}}{2} \right\} \\
& \times \prod_{t=\ell_{k+1}+h'_{k+1}+1}^{\ell_k+h_k} \frac{\sqrt{\lambda_k \lambda_{k+2}}^{\kappa_t}}{\kappa_t!} \exp\left\{ -\frac{\lambda_k + \lambda_{k+2}}{2} \right\} \\
& \times \prod_{t=\ell_k+h_k+1}^{\ell_{k+1}} \frac{\sqrt{\lambda_{k+1} \lambda_{k+2}}^{\kappa_t}}{\kappa_t!} \exp\left\{ -\frac{\lambda_{k+1} + \lambda_{k+2}}{2} \right\} \Bigg) \\
= \frac{\rho^{(\ell_{k+1}+h'_{k+1})-\ell_k}(\lambda_k, \lambda_{k+1}) \rho^{\ell_{k+1}-(\ell_k+h_k)}(\lambda_{k+1}, \lambda_{k+2})}{\rho^{(\ell_{k+1}+h'_{k+1})-(\ell_k+h_k)}(\lambda_k, \lambda_{k+2})} &
\end{aligned} \tag{54}$$

using the same manipulations as those leading to (48). Note that in this case, $\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_{k+1})$ effectively depends on $\boldsymbol{\ell}$. We introduce the following definition, where the notation $\boldsymbol{\lambda}_{k:k+2}$ denotes the truncated vector $[\lambda_k, \lambda_{k+1}, \lambda_{k+2}]^T$ of $\boldsymbol{\lambda}$

$$r(\boldsymbol{\lambda}_{k:k+2}) = \frac{\rho(\lambda_k, \lambda_{k+1})\rho(\lambda_{k+1}, \lambda_{k+2})}{\rho(\lambda_k, \lambda_{k+2})} \tag{55}$$

$$= \exp\left\{ -\lambda_{k+1} + \sqrt{\lambda_k \lambda_{k+1}} + \sqrt{\lambda_{k+1} \lambda_{k+2}} - \sqrt{\lambda_k \lambda_{k+2}} \right\}. \tag{56}$$

Then we retrieve the function $r(.)$ used in (23). Using this definition, we can rewrite (54) as

$$\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_{k+1}) = \rho^{h_k}(\lambda_k, \lambda_{k+1})\, \rho^{-h'_{k+1}}(\lambda_{k+1}, \lambda_{k+2})\, r^{(\ell_{k+1}+h'_{k+1})-(\ell_k+h_k)}(\boldsymbol{\lambda}_{k:k+2}) \tag{57}$$

iii) Case D ($l = k$): In this case, we have to consider the fact that $\boldsymbol{h}'_k$ can either take three values: either $\boldsymbol{h}'_k = \boldsymbol{h}_k$, or $\boldsymbol{h}'_k = -\boldsymbol{h}_k$, or $\boldsymbol{h}'_k = \boldsymbol{0}_{2K+1}$, according to (34).

If $\boldsymbol{h}'_k = \boldsymbol{h}_k$, (38) gives directly

$$\begin{aligned}
\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}_k) &= \sum_{\boldsymbol{\kappa}\in\mathbb{N}^T} f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_k) \\
&= 1
\end{aligned} \tag{58}$$

since it is the sum of a probability distribution over its whole domain.

If $\boldsymbol{h}'_k = -\boldsymbol{h}_k$, equation (46) becomes

$$\begin{aligned}
\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_l) = \sum_{\kappa_1=0}^{+\infty} \cdots \sum_{\kappa_T=0}^{+\infty} \Bigg(& \prod_{\substack{i=1\\ i\neq k,k+1}}^{K+1} \left[\prod_{t=\ell_{i-1}+1}^{\ell_i} \frac{\lambda_i^{\kappa_t}}{\kappa_t!} \exp(-\lambda_i) \right] \prod_{t=\ell_{k-1}+1}^{\ell_k-|h_k|} \frac{\lambda_k^{\kappa_t}}{\kappa_t!} \exp(-\lambda_k) \\
& \times \prod_{t=\ell_k-|h_k|+1}^{\ell_k+|h_k|} \frac{\sqrt{\lambda_k \lambda_{k+1}}^{\kappa_t}}{\kappa_t!} \exp\left\{ -\frac{\lambda_k + \lambda_{k+1}}{2} \right\} \\
& \times \prod_{t=\ell_k+|h_k|+1}^{\ell_{k+1}} \frac{\lambda_{k+1}^{\kappa_t}}{\kappa_t!} \exp(-\lambda_{k+1}) \Bigg) \\
= \rho^{2|h_k|}(\lambda_k, \lambda_{k+1}) &
\end{aligned} \tag{59}$$

using the same manipulations as those leading to (48).

At last, if $\boldsymbol{h}_k' = \boldsymbol{0}_{2K+1}$, then equation (46) becomes:

$$\begin{aligned}
\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{0}_{2K+1}) &= \sum_{\kappa_1=0}^{+\infty} \cdots \sum_{\kappa_T=0}^{+\infty} \left(\prod_{\substack{i=1 \\ i\neq k,k+1}}^{K+1} \left[\prod_{t=\ell_{i-1}+1}^{\ell_i} \frac{\lambda_i^{\kappa_t}}{\kappa_t!} \exp(-\lambda_i) \right] \prod_{t=\ell_{k-1}+1}^{\ell_k - \max(-h_k,0)} \frac{\lambda_k^{\kappa_t}}{\kappa_t!} \exp(-\lambda_k) \right. \\
&\qquad \times \prod_{t=\ell_k-\max(-h_k,0)+1}^{\ell_k+\max(h_k,0)} \frac{\sqrt{\lambda_k \lambda_{k+1}}^{\kappa_t}}{\kappa_t!} \exp\left\{ -\frac{\lambda_k + \lambda_{k+1}}{2} \right\} \\
&\qquad \left. \times \prod_{t=\ell_k+\max(h_k,0)+1}^{\ell_{k+1}} \frac{\lambda_{k+1}^{\kappa_t}}{\kappa_t!} \exp(-\lambda_{k+1}) \right) \\
&= \rho^{|h_k|}(\lambda_k, \lambda_{k+1})
\end{aligned} \tag{60}$$

using once again the same manipulations as those leading to (48).

3) Derivation of $\zeta(\boldsymbol{h}_k, \boldsymbol{h}_l')$ and final expressions of $\boldsymbol{V}_{22}$ and $\boldsymbol{P}_{22}$: In order to obtain closed-form expressions of $\zeta(\boldsymbol{h}_k, \boldsymbol{h}_l')$, we use the expressions obtained for $\pi(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}_l')$ and $\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}_l')$ in the three cases

- *i)* UT, using (39) with $\mathcal{J}_{h_k,h_l'}$ given by (40), and (48),
- *ii)* FSD, using (39) with $\mathcal{J}_{h_k,h_{k+1}'}$ given by (41), and either (54) or (53), depending on whether there is overlap or not, respectively (see condition (52)),
- *iii)* D, using (39), with $\mathcal{J}_{h_k,h_k'}$ given by (42), and either (58), (59) or (60) depending on whether $h_k' = h_k$, $h_k' = -h_k$ or $h_k' = 0$.

We then plug them into (36) to obtain the final expression of $\zeta(\boldsymbol{h}_k, \boldsymbol{h}_l')$. Finally, the expressions of $\boldsymbol{V}_{22}$ and $\boldsymbol{P}_{22}$ are obtained by using (34). Let us give these details in each of the three aforementioned cases UT, FSD and D.

i) Case UT ($l > k+1$) and derivation of the upper-triangle terms of $\boldsymbol{P}_{22}$: Notice first that this case enable us to derive the upper-triangle terms of $\boldsymbol{P}_{22}$, according to (34). As just explained, by plugging (39) (in which $\mathcal{J}_{h_k,h_l'}$ is given by (40)) and (48) into (36), we obtain

$$\begin{aligned}
\zeta(\boldsymbol{h}_k, \boldsymbol{h}_l') &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^K} \int_{\mathbb{R}_+^{K+1}} \left(\prod_{i=1}^{K+1} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i-1} \exp\left(-\beta\lambda_i\right) \right) \frac{1}{\tau^K} \mathbb{I}_{\mathcal{J}_{h_k,h_l'}}(\boldsymbol{\ell})\, \rho^{|h_k|}(\lambda_k, \lambda_{k+1}) \rho^{|h_l'|}(\lambda_l, \lambda_{l+1}) \mathrm{d}\boldsymbol{\lambda} \\
&= \frac{1}{\tau^K} \left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}^K} \mathbb{I}_{\mathcal{J}_{h_k,h_l'}}(\boldsymbol{\ell}) \right) \frac{\beta^{\alpha_k+\alpha_{k+1}}}{\Gamma(\alpha_k)\Gamma(\alpha_{k+1})} \frac{\beta^{\alpha_l+\alpha_{l+1}}}{\Gamma(\alpha_l)\Gamma(\alpha_{l+1})} \\
&\quad \times \left(\int_{\mathbb{R}_+^2} \lambda_k^{\alpha_k-1} \lambda_{k+1}^{\alpha_{k+1}-1} \exp\left\{ -\beta\left(\lambda_k+\lambda_{k+1}\right) - |h_k| \frac{\left(\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k}\right)^2}{2} \right\} \mathrm{d}\lambda_k \mathrm{d}\lambda_{k+1} \right) \\
&\quad \times \left(\int_{\mathbb{R}_+^2} \lambda_l^{\alpha_l-1} \lambda_{l+1}^{\alpha_{l+1}-1} \exp\left\{ -\beta\left(\lambda_l+\lambda_{l+1}\right) - |h_l'| \frac{\left(\sqrt{\lambda_{l+1}} - \sqrt{\lambda_l}\right)^2}{2} \right\} \mathrm{d}\lambda_l \mathrm{d}\lambda_{l+1} \right) \\
&= \frac{\mathrm{Card}\left(\mathcal{J}_{h_k,h_l'}\right)}{\tau^K} \Phi\left(h_k\right) \Phi\left(h_l'\right)
\end{aligned} \tag{61}$$

in which $\Phi\left(.\right)$ is defined in (19), and $\mathrm{Card}\left(\mathcal{J}_{h_k,h_l'}\right)$ denotes the cardinality of the set $\mathcal{J}_{h_k,h_l'}$. By analyzing Table I, we can show that

$$\begin{aligned}
\frac{\mathrm{Card}\left(\mathcal{J}_{h_k,h_l'}\right)}{\tau^K} &= u\left(\tau, h_k\right) u\left(\tau, h_l\right) \\
&= \frac{\mathrm{Card}\left(\mathcal{J}_{-h_k,-h_l}\right)}{\tau^K} = \frac{\mathrm{Card}\left(\mathcal{J}_{-h_k,h_l}\right)}{\tau^K} = \frac{\mathrm{Card}\left(\mathcal{J}_{h_k,-h_l}\right)}{\tau^K}
\end{aligned} \tag{62}$$

where $u\left(\tau, .\right)$ is defined in (21). Then, plugging (61) into (34), we obtain

$$\begin{aligned}
\left[\boldsymbol{P}_{22}\right]_{k,l} &= \zeta\left(\boldsymbol{h}_k, \boldsymbol{h}_l\right) + \zeta\left(-\boldsymbol{h}_k, -\boldsymbol{h}_l\right) - \zeta\left(-\boldsymbol{h}_k, \boldsymbol{h}_l\right) - \zeta\left(\boldsymbol{h}_k, -\boldsymbol{h}_l\right) \\
&= 0
\end{aligned} \tag{63}$$

since neither $\Phi\left(h_k\right)$ nor $\mathrm{Card}\left(\mathcal{J}_{h_k,h_l'}\right)$ depend on the signs of h_k and h_l'. Equation (63) finally gives us the tridiagonal structure of matrix $\boldsymbol{P}_{22}$, that appears in (28).

ii) Case FSD ($l = k+1$) and derivation of the first superdiagonal terms of $\boldsymbol{P}_{22}$: Let us first remark that the expression of $[\boldsymbol{P}_{22}]_{k,k+1}$ in (34) can be rewritten as

$$[\boldsymbol{P}_{22}]_{k,k+1} = \mathrm{sign}(h_k h_{k+1})\big[\zeta(|h_k|, |h_{k+1}|) + \zeta(-|h_k|, -|h_{k+1}|) - \zeta(-|h_k|, |h_{k+1}|) - \zeta(|h_k|, -|h_{k+1}|)\big]. \tag{64}$$

This writing enables us to dispose of considerations on the signs of h_k and h_{k+1} in order to determine in which term of (34), a possible overlap has to be taken into account. Here, this has to be done only for the last term $\zeta(|h_k|, -|h_{k+1}|)$, whatever the signs of h_k and h_{k+1}.

Let us first derive $\zeta(h_k, h'_{k+1})$ in the case without overlap, i.e., when the condition (52) is *not* met. This will give us the expression for the three first terms of (64), and possibly the fourth if $\min(|h_k|, |h_{k+1}|) = 1$. As already explained, by plugging (39) (in which $\mathcal{J}_{h_k,h'_l}$ is given by (41)) and (53) into (36), we obtain, in the same way as in the case UT:

$$\begin{aligned}
\zeta(\boldsymbol{h}_k, \boldsymbol{h}'_{k+1}) &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^K} \int_{\mathbb{R}_+^{K+1}} \left(\prod_{i=1}^{K+1} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} \exp(-\beta\lambda_i)\right) \frac{1}{\tau^K} \mathbb{I}_{\mathcal{J}_{h_k,h'_{k+1}}}(\boldsymbol{\ell})\, \rho^{|h_k|}(\lambda_k, \lambda_{k+1}) \rho^{|h'_{k+1}|}(\lambda_{k+1}, \lambda_{k+2})\, \mathrm{d}\boldsymbol{\lambda} \\
&= \frac{1}{\tau^K} \left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}^K} \mathbb{I}_{\mathcal{J}_{h_k,h'_{k+1}}}(\boldsymbol{\ell})\right) \frac{\beta^{\alpha_k + \alpha_{k+1} + \alpha_{k+2}}}{\Gamma(\alpha_k)\Gamma(\alpha_{k+1})\Gamma(\alpha_{k+2})} \\
&\quad \times \int_{\mathbb{R}_+^3} \lambda_k^{\alpha_k - 1} \lambda_{k+1}^{\alpha_{k+1}-1} \lambda_{k+2}^{\alpha_{k+2}-1} \exp\Bigg\{ -\beta(\lambda_k + \lambda_{k+1} + \lambda_{k+2}) - |h_k| \frac{(\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k})^2}{2} \\
&\qquad\qquad - \left|h'_{k+1}\right| \frac{(\sqrt{\lambda_{k+2}} - \sqrt{\lambda_{k+1}})^2}{2} \Bigg\} \mathrm{d}\lambda_k\, \mathrm{d}\lambda_{k+1}\, \mathrm{d}\lambda_{k+2} \\
&= \frac{\mathrm{Card}\left(\mathcal{J}_{h_k,h'_{k+1}}\right)}{\tau^K} \frac{\beta^{\alpha_k + \alpha_{k+1} + \alpha_{k+2}}}{\Gamma(\alpha_k)\Gamma(\alpha_{k+1})\Gamma(\alpha_{k+2})} \int_{\mathbb{R}_+^3} \phi_{h_k,h'_{k+1}}(\boldsymbol{\lambda}_{k:k+2})\, \mathrm{d}\lambda_k\, \mathrm{d}\lambda_{k+1}\, \mathrm{d}\lambda_{k+2}
\end{aligned} \tag{65}$$

where $\phi_{h_k,h'_{k+1}}(.)$ is defined in (20). By reading Table II, we can deduce that

$$\frac{\mathrm{Card}\left(\mathcal{J}_{h_k,h'_{k+1}}\right)}{\tau^K} = \begin{cases} \dfrac{(\tau - |h_k|)\big(\tau - \max(-h_k, h'_{k+1}, 0) - \max(h_k, -h'_{k+1}, 0)\big)(\tau - \left|h'_{k+1}\right|)}{\tau^3}, & \text{if } k+1 \leq K-1 \text{ and } \max(|h_k|, \left|h'_{k+1}\right|) \leq \tau - 1, \\ \dfrac{(\tau - |h_{K-1}|)\big(\tau - \max(-h_{K-1}, h'_K, 0) - \max(h_{K-1}, -h'_K, 0)\big)}{\tau^2}, & \text{if } k+1 = K \text{ and } \max(|h_{K-1}|, |h'_K|) \leq \tau - 1, \\ 0, & \text{if } \max(|h_{K-1}|, |h'_K|) \geq \tau. \end{cases} \tag{66}$$

Finally, the results for the first three terms in (64) are, for $k < K-1$:

$$\begin{aligned}
\zeta(|h_k|, |h_{k+1}|) = \zeta(-|h_k|, -|h_{k+1}|) &= \frac{(\tau - |h_k|)(\tau - |h_k| - |h_{k+1}|)(\tau - |h_{k+1}|)}{\tau^3} \\
&\quad \times \frac{\beta^{\alpha_k + \alpha_{k+1} + \alpha_{k+2}}}{\Gamma(\alpha_k)\Gamma(\alpha_{k+1})\Gamma(\alpha_{k+2})} \int_{\mathbb{R}_+^3} \phi_{|h_k|,|h_{k+1}|}(\boldsymbol{\lambda}_{k:k+2})\, \mathrm{d}\boldsymbol{\lambda}_{k:k+2}
\end{aligned} \tag{67}$$

$$\begin{aligned}
\zeta(-|h_k|, |h_{k+1}|) &= \frac{(\tau - |h_k|)\big(\tau - \max(|h_k|, |h_{k+1}|)\big)(\tau - |h_{k+1}|)}{\tau^3} \\
&\quad \times \frac{\beta^{\alpha_k + \alpha_{k+1} + \alpha_{k+2}}}{\Gamma(\alpha_k)\Gamma(\alpha_{k+1})\Gamma(\alpha_{k+2})} \int_{\mathbb{R}_+^3} \phi_{|h_k|,|h_{k+1}|}(\boldsymbol{\lambda}_{k:k+2})\, \mathrm{d}\boldsymbol{\lambda}_{k:k+2}
\end{aligned} \tag{68}$$

and for $k = K-1$, (67) and (68) become respectively

$$\begin{aligned}
\zeta(|h_{K-1}|, |h_K|) = \zeta(-|h_{K-1}|, -|h_K|) &= \frac{(\tau - |h_{K-1}|)(\tau - |h_{K-1}| - |h_K|)}{\tau^2} \\
&\quad \times \frac{\beta^{\alpha_{K-1} + \alpha_K + \alpha_{K+1}}}{\Gamma(\alpha_{K-1})\Gamma(\alpha_K)\Gamma(\alpha_{K+1})} \int_{\mathbb{R}_+^3} \phi_{|h_{K-1}|,|h_K|}(\boldsymbol{\lambda}_{K-1:K+1})\, \mathrm{d}\boldsymbol{\lambda}_{K-1:K+1}
\end{aligned} \tag{69}$$

and

$$\zeta(-|h_{K-1}|,|h_K|) = \frac{(\tau - |h_{K-1}|)\big(\tau - \max(|h_{K-1}|,|h_K|)\big)}{\tau^2} \times \frac{\beta^{\alpha_{K-1}+\alpha_K+\alpha_{K+1}}}{\Gamma(\alpha_{K-1})\Gamma(\alpha_K)\Gamma(\alpha_{K+1})} \int_{\mathbb{R}_+^3} \phi_{|h_{K-1}|,|h_K|}(\boldsymbol{\lambda}_{K-1:K+1})\,\mathrm{d}\boldsymbol{\lambda}_{K-1:K+1} \tag{70}$$

If there is no overlap (i.e., $\min(|h_k|, |h_{k+1}|) = 1$), the fourth term in (64) is also given by (68) or (70), i.e., we have

$$\zeta(|h_k|, -|h_{k+1}|) = \zeta(-|h_k|, |h_{k+1}|). \tag{71}$$

This term has a different writing if there is an overlap, i.e., when $\min(|h_k|, |h_{k+1}|) > 1$. By referring to the overlap condition (50), it appears that for values of ℓ_{k+1} that satisfy $\ell_{k+1} < \ell_k + |h_k| + |h_{k+1}|$, the quantity $\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, |\boldsymbol{h}_k|, -|\boldsymbol{h}_{k+1}|)$ has to be written according to (54), whereas for values of ℓ_{k+1} such that $\ell_{k+1} \geq \ell_k + |h_k| + |h_{k+1}|$, the quantity $\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, |\boldsymbol{h}_k|, -|\boldsymbol{h}_{k+1}|)$ has to be written according to (53). Then, the sum in (36) w.r.t. ℓ_{k+1} has to be split into two parts, the first of which contains the terms with overlap, i.e., for $\ell_{k+1} \in \{\ell_k + \max(|h_k|, |h_{k+1}|) + 1, \ldots, \ell_k + |h_k| + |h_{k+1}| - 1\}$, whereas the second part contains the terms without overlap, i.e., for $\ell_{k+1} \in \{\ell_k + |h_k| + |h_{k+1}|, \ldots, \ell_k + \tau\}$. More explicitly, we rewrite (36) in the following way

$$\begin{aligned}\zeta(|h_k|, -|h_{k+1}|) &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^K} \int_{\mathbb{R}_+^{K+1}} \pi(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_{k+1})\; \acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_k, \boldsymbol{h}'_{k+1})\,\mathrm{d}\boldsymbol{\lambda} \\ &= \int_{\mathbb{R}_+^{K+1}} \left[\prod_{i=1}^{K+1} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i-1} \exp(-\beta\lambda_i) \left(\sum_{\boldsymbol{\ell}_{1:k}\in\mathbb{Z}^k} \big[S_1(\boldsymbol{\lambda}, \boldsymbol{\ell}_{1:k}, h_k, h_{k+1}) + S_2(\boldsymbol{\lambda}, \boldsymbol{\ell}_{1:k}, h_k, h_{k+1})\big]\right)\right] \mathrm{d}\boldsymbol{\lambda}\end{aligned} \tag{72}$$

in which, for $i, j = 1, \ldots, K$, $\boldsymbol{\ell}_{i:j}$ denotes the truncated vector $[\ell_i, \ldots, \ell_j]^T$, the quantity $S_1(\boldsymbol{\lambda}, \boldsymbol{\ell}_{1:k}, h_k, h_{k+1})$ is

$$S_1(\boldsymbol{\lambda}, \boldsymbol{\ell}_{1:k}, h_k, h_{k+1}) = \sum_{\ell_{k+1}=(\ell_{k+1})_{\min}}^{\ell_k+|h_k|+|h_{k+1}|-1} \left[\sum_{\boldsymbol{\ell}_{k+2:K}\in\mathbb{Z}^{K-k-1}} \frac{1}{\tau^K} \mathbb{I}_{\mathcal{J}_{|h_k|,-|h_{k+1}|}}(\boldsymbol{\ell})\; \acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, |h_k|, -|h_{k+1}|)\right] \tag{73}$$

in which $\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, |h_k|, -|h_{k+1}|)$ is expressed according to (57), and $S_2(\boldsymbol{\lambda}, \boldsymbol{\ell}_{1:k}, h_k, h_{k+1})$ denotes

$$S_2(\boldsymbol{\lambda}, \boldsymbol{\ell}_{1:k}, h_k, h_{k+1}) = \sum_{\ell_{k+1}=\ell_k+|h_k|+|h_{k+1}|}^{\ell_k+\tau} \left[\sum_{\boldsymbol{\ell}_{k+2:K}\in\mathbb{Z}^{K-k-1}} \frac{1}{\tau^K} \mathbb{I}_{\mathcal{J}_{|h_k|,-|h_{k+1}|}}(\boldsymbol{\ell})\; \acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, |h_k|, -|h_{k+1}|)\right] \tag{74}$$

in which $\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, |h_k|, -|h_{k+1}|)$ is expressed according to (53).

For the subsequent derivation, we will use the following rewriting of $\mathcal{J}_{|h_k|,-|h_{k+1}|}$, for $k < K-1$:

$$\mathcal{J}_{|h_k|,-|h_{k+1}|} = \tilde{\mathcal{J}}_{1:k,|h_k|} \times \{\ell_k + \max(|h_k|, |h_{k+1}|) + 1, \ldots, \ell_k + \tau\} \times \tilde{\mathcal{J}}_{k+2:K,-|h_{k+1}|} \tag{75}$$

with

$$\tilde{\mathcal{J}}_{1:k,|h_k|} = \left(\prod_{i=1}^{k-1} \{\ell_{i-1}+1, \ldots, \ell_{i-1}+\tau\}\right) \times \{\ell_{k-1}+1, \ldots, \ell_{k-1} - |h_k| + \tau\} \tag{76}$$

and

$$\tilde{\mathcal{J}}_{k+2:K,-|h_{k+1}|} = \{\ell_{k+1}+1, \ldots, \ell_{k+1} - |h_{k+1}| + \tau\} \times \left(\prod_{i=k+3}^{K} \{\ell_{i-1}+1, \ldots, \ell_{i-1}+\tau\}\right). \tag{77}$$

In the same manner, for $k = K-1$, we will rewrite $\mathcal{J}_{|h_k|,-|h_{k+1}|}$ as

$$\mathcal{J}_{|h_{K-1}|,-|h_K|} = \tilde{\mathcal{J}}_{1:K-1,|h_{K-1}|} \times \{\ell_{K-1} + \max(|h_{K-1}|, |h_K|) + 1, \ldots, \ell_{K-1} + \tau\}. \tag{78}$$

Let us first develop (73), for $k < K-1$, by splitting $\mathcal{J}_{|h_k|,-|h_{k+1}|}$ according to (75)

$$
\begin{aligned}
S_1(\boldsymbol{\lambda}, \boldsymbol{\ell}_{1:k}, h_k, h_{k+1}) &= \frac{1}{\tau^K} \frac{\rho^{|h_k|}(\lambda_k, \lambda_{k+1})\, \rho^{|h_{k+1}|}(\lambda_{k+1}, \lambda_{k+2})}{\big(r(\boldsymbol{\lambda}_{k:k+2})\big)^{\ell_k + |h_{k+1}| + |h_k|}} \\
&\quad \times \sum_{\ell_{k+1} = \ell_k + \max(|h_k|, |h_{k+1}|)+1}^{\ell_k + |h_k| + |h_{k+1}| - 1} \left(\big(r(\boldsymbol{\lambda}_{k:k+2})\big)^{\ell_{k+1}} \sum_{\boldsymbol{\ell}_{k+2:K} \in \mathbb{Z}^{K-k-1}} \mathbb{I}_{\mathcal{J}_{|h_k|,-|h_{k+1}|}}(\boldsymbol{\ell}) \right) \\
&= \frac{\mathrm{Card}\left(\tilde{\mathcal{J}}_{k+2:K,-|h_{k+1}|}\right)}{\tau^K} \frac{\rho^{|h_k|}(\lambda_k, \lambda_{k+1})\, \rho^{|h_{k+1}|}(\lambda_{k+1}, \lambda_{k+2})}{\big(r(\boldsymbol{\lambda}_{k:k+2})\big)^{\ell_k + |h_{k+1}| + |h_k|}} \mathbb{I}_{\tilde{\mathcal{J}}_{1:k,|h_k|}}(\boldsymbol{\ell}_{1:k}) \\
&\quad \times \sum_{\ell_{k+1} = \ell_k + \max(|h_k|, |h_{k+1}|)+1}^{\ell_k + |h_k| + |h_{k+1}| - 1} \big(r(\boldsymbol{\lambda}_{k:k+2})\big)^{\ell_{k+1}} \\
&= \frac{\mathrm{Card}\left(\tilde{\mathcal{J}}_{k+2:K,-|h_{k+1}|}\right)}{\tau^K} \frac{\rho^{|h_k|}(\lambda_k, \lambda_{k+1})\, \rho^{|h_{k+1}|}(\lambda_{k+1}, \lambda_{k+2})}{\big(r(\boldsymbol{\lambda}_{k:k+2})\big)^{\ell_k + |h_{k+1}| + |h_k|}} \mathbb{I}_{\tilde{\mathcal{J}}_{1:k,|h_k|}}(\boldsymbol{\ell}_{1:k}) \, \big(r(\boldsymbol{\lambda}_{k:k+2})\big)^{\ell_k + \max(|h_k|,|h_{k+1}|)+1} \\
&\quad \times \frac{1 - \big(r(\boldsymbol{\lambda}_{k:k+2})\big)^{|h_k| + |h_{k+1}| - \max(|h_k|,|h_{k+1}|) - 1}}{1 - r(\boldsymbol{\lambda}_{k:k+2})}, \qquad \text{(by summing the terms of the geometric series with common ratio } r(\boldsymbol{\lambda}_{k:k+2})\text{)} \\
&= \frac{\mathrm{Card}\left(\tilde{\mathcal{J}}_{k+2:K,-|h_{k+1}|}\right)}{\tau^K} \rho^{|h_k|}(\lambda_k, \lambda_{k+1})\, \rho^{|h_{k+1}|}(\lambda_{k+1}, \lambda_{k+2}) \big(r(\boldsymbol{\lambda}_{k:k+2})\big)^{1 - \min(|h_k|,|h_{k+1}|)} \\
&\quad \times \frac{1 - \big(r(\boldsymbol{\lambda}_{k:k+2})\big)^{\min(|h_k|,|h_{k+1}|) - 1}}{1 - r(\boldsymbol{\lambda}_{k:k+2})} \mathbb{I}_{\tilde{\mathcal{J}}_{1:k,|h_k|}}(\boldsymbol{\ell}_{1:k}).
\end{aligned} \tag{79}
$$

According to (77), we have

$$
\mathrm{Card}\left(\tilde{\mathcal{J}}_{k+2:K,-|h_{k+1}|}\right) = \begin{cases} (\tau - |h_{k+1}|)\tau^{K-k-2}, & \text{if } |h_{k+1}| \leq \tau - 1 \\ 0, & \text{if } |h_{k+1}| \geq \tau \end{cases} \tag{80}
$$

so S_1 is given, for $k < K-1$, and provided that $|h_{k+1}| \leq \tau - 1$, by

$$
S_1(\boldsymbol{\lambda}, \boldsymbol{\ell}_{1:k}, h_k, h_{k+1}) = \frac{(\tau - |h_{k+1}|)}{\tau^{k+2}} \rho^{|h_k|}(\lambda_k, \lambda_{k+1})\, \rho^{|h_{k+1}|}(\lambda_{k+1}, \lambda_{k+2}) \frac{\big(r(\boldsymbol{\lambda}_{k:k+2})\big)^{1 - \min(|h_k|,|h_{k+1}|)} - 1}{1 - r(\boldsymbol{\lambda}_{k:k+2})} \mathbb{I}_{\tilde{\mathcal{J}}_{1:k,|h_k|}}(\boldsymbol{\ell}_{1:k}) \tag{81}
$$

and if $k = K-1$, by splitting $\mathcal{J}_{|h_{K-1}|,-|h_K|}$ according to (78), we obtain, by similar manipulations as those leading to (79)

$$
\begin{aligned}
S_1(\boldsymbol{\lambda}, \boldsymbol{\ell}_{1:K-1}, h_{K-1}, h_K) &= \frac{1}{\tau^K} \rho^{|h_{K-1}|}(\lambda_{K-1}, \lambda_K)\, \rho^{|h_K|}(\lambda_K, \lambda_{K+1}) \frac{\big(r(\boldsymbol{\lambda}_{K-1:K+1})\big)^{1 - \min(|h_{K-1}|,|h_K|)} - 1}{1 - r(\boldsymbol{\lambda}_{K-1:K+1})} \\
&\quad \times \mathbb{I}_{\tilde{\mathcal{J}}_{1:K-1,|h_{K-1}|}}(\boldsymbol{\ell}_{1:K-1}).
\end{aligned} \tag{82}
$$

Let us now develop S_2 from (74) where, as already mentioned, $\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, |h_k|, -|h_{k+1}|)$ is given by (53). For $k < K-1$,

$$
\begin{aligned}
S_2(\boldsymbol{\lambda}, \boldsymbol{\ell}_{1:k}, h_k, h_{k+1}) &= \sum_{\ell_{k+1} = \ell_k + |h_k| + |h_{k+1}|}^{\ell_k + \tau} \left[\sum_{\boldsymbol{\ell}_{k+2:K} \in \mathbb{Z}^{K-k-1}} \frac{1}{\tau^K} \mathbb{I}_{\mathcal{J}_{|h_k|,-|h_{k+1}|}}(\boldsymbol{\ell})\, \rho^{|h_k|}(\lambda_k, \lambda_{k+1}) \rho^{|h_{k+1}|}(\lambda_{k+1}, \lambda_{k+2}) \right] \\
&= \frac{1}{\tau^K} \rho^{|h_k|}(\lambda_k, \lambda_{k+1}) \rho^{|h_{k+1}|}(\lambda_{k+1}, \lambda_{k+2}) \sum_{\ell_{k+1} = \ell_k + |h_k| + |h_{k+1}|}^{\ell_k + \tau} \left[\sum_{\boldsymbol{\ell}_{k+2:K} \in \mathbb{Z}^{K-k-1}} \mathbb{I}_{\mathcal{J}_{|h_k|,-|h_{k+1}|}}(\boldsymbol{\ell}) \right] \\
&= \frac{\mathrm{Card}\left(\tilde{\mathcal{J}}_{k+2:K,-|h_{k+1}|}\right) \max\big(\tau - |h_k| - |h_{k+1}| + 1, 0\big)}{\tau^K} \rho^{|h_k|}(\lambda_k, \lambda_{k+1}) \rho^{|h_{k+1}|}(\lambda_{k+1}, \lambda_{k+2}) \mathbb{I}_{\tilde{\mathcal{J}}_{1:k,|h_k|}}(\boldsymbol{\ell}_{1:k}) \\
&= \frac{\big(\tau - |h_{k+1}|\big)\big(\tau - |h_k| - |h_{k+1}| + 1\big)}{\tau^{k+2}} \rho^{|h_k|}(\lambda_k, \lambda_{k+1}) \rho^{|h_{k+1}|}(\lambda_{k+1}, \lambda_{k+2}) \mathbb{I}_{\tilde{\mathcal{J}}_{1:k,|h_k|}}(\boldsymbol{\ell}_{1:k}),
\end{aligned} \tag{83}
$$

provided that $\tau - |h_k| - |h_{k+1}| + 1 > 0$ and $|h_{k+1}| \leq \tau - 1$.

If $k = K-1$, we obtain in the same way

$$
S_2(\boldsymbol{\lambda}, \boldsymbol{\ell}_{1:K-1}, h_{K-1}, h_K) = \frac{\max\big(\tau - |h_{K-1}| - |h_K| + 1, 0\big)}{\tau^K} \rho^{|h_{K-1}|}(\lambda_{K-1}, \lambda_K) \rho^{|h_K|}(\lambda_K, \lambda_{K+1}) \mathbb{I}_{\tilde{\mathcal{J}}_{1:K-1,|h_{K-1}|}}(\boldsymbol{\ell}_{1:K-1}). \tag{84}
$$

We then plug (81) and (83) into (72) to obtain, for $k < K-1$

$$\zeta(|h_k|, -|h_{k+1}|) = \int_{\mathbb{R}_+^{K+1}} \left[\prod_{i=1}^{K+1} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} \exp(-\beta\lambda_i) \left(\sum_{\boldsymbol{\ell}_{1:k} \in \mathbb{Z}^k} \frac{(\tau - |h_{k+1}|)}{\tau^{k+2}} \rho^{|h_k|}(\lambda_k, \lambda_{k+1}) \rho^{|h_{k+1}|}(\lambda_{k+1}, \lambda_{k+2}) \right.\right.$$
$$\left.\left. \times \left(\frac{\left(r(\boldsymbol{\lambda}_{k:k+2})\right)^{1-\min(|h_k|,|h_{k+1}|)} - 1}{1 - r(\boldsymbol{\lambda}_{k:k+2})} + \max(\tau - |h_k| - |h_{k+1}| + 1, 0) \right) \mathbb{I}_{\tilde{\mathcal{J}}_{1:k,|h_k|}}(\boldsymbol{\ell}_{1:k}) \right) \right] \mathrm{d}\boldsymbol{\lambda}$$
$$= \frac{\operatorname{Card}\left(\tilde{\mathcal{J}}_{k+2:K,-|h_{k+1}|}\right)}{\tau^K} \int_{\mathbb{R}_+^{K+1}} \left[\prod_{i=1}^{K+1} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} \exp(-\beta\lambda_i) \rho^{|h_k|}(\lambda_k, \lambda_{k+1}) \rho^{|h_{k+1}|}(\lambda_{k+1}, \lambda_{k+2}) \right.$$
$$\times \left(\frac{\left(r(\boldsymbol{\lambda}_{k:k+2})\right)^{1-\min(|h_k|,|h_{k+1}|)} - 1}{1 - r(\boldsymbol{\lambda}_{k:k+2})} + \max(\tau - |h_k| - |h_{k+1}| + 1, 0) \right)$$
$$\left. \times \sum_{\boldsymbol{\ell}_{1:k} \in \mathbb{Z}^k} \mathbb{I}_{\tilde{\mathcal{J}}_{1:k,|h_k|}}(\boldsymbol{\ell}_{1:k}) \right] \mathrm{d}\boldsymbol{\lambda}$$
$$= \frac{\operatorname{Card}\left(\tilde{\mathcal{J}}_{k+2:K,-|h_{k+1}|}\right) \operatorname{Card}\left(\tilde{\mathcal{J}}_{1:k,|h_k|}\right)}{\tau^K}$$
$$\times \int_{\mathbb{R}_+^{K+1}} \left[\prod_{i=1}^{K+1} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} \exp(-\beta\lambda_i) \rho^{|h_k|}(\lambda_k, \lambda_{k+1}) \rho^{|h_{k+1}|}(\lambda_{k+1}, \lambda_{k+2}) \right.$$
$$\left. \times \left(\frac{\left(r(\boldsymbol{\lambda}_{k:k+2})\right)^{1-\min(|h_k|,|h_{k+1}|)} - 1}{1 - r(\boldsymbol{\lambda}_{k:k+2})} + \max\bigl(\tau - |h_k| - |h_{k+1}| + 1, 0\bigr) \right) \right] \mathrm{d}\boldsymbol{\lambda} \tag{85}$$

where $\operatorname{Card}\left(\tilde{\mathcal{J}}_{1:k,|h_k|}\right)$ is given, according to (76), by

$$\operatorname{Card}\left(\tilde{\mathcal{J}}_{1:k,|h_k|}\right) = \begin{cases} \tau^{k-1}\bigl(\tau - |h_k|\bigr), & \text{if } |h_k| \leq \tau - 1; \\ 0, & \text{if } |h_k| \geq \tau. \end{cases} \tag{86}$$

Then, after integrating out w.r.t. variables $\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+3}, \ldots, \lambda_K$ in (85), and provided that $\max\bigl(|h_k|, |h_{k+1}|\bigr) \leq \tau - 1$, we finally obtain

$$\zeta(|h_k|, -|h_{k+1}|) = \frac{\bigl(\tau - |h_k|\bigr)\bigl(\tau - |h_{k+1}|\bigr)}{\tau^3} \int_{\mathbb{R}_+^3} \phi_{|h_k|,|h_{k+1}|}(\boldsymbol{\lambda}_{k:k+2}) \left(\frac{\left(r(\boldsymbol{\lambda}_{k:k+2})\right)^{1-\min(|h_k|,|h_{k+1}|)} - 1}{1 - r(\boldsymbol{\lambda}_{k:k+2})} \right.$$
$$\left. + \max\bigl(\tau - |h_k| - |h_{k+1}| + 1, 0\bigr) \right) \mathrm{d}\boldsymbol{\lambda}_{k:k+2} \tag{87}$$

and for $k = K-1$, by plugging (82) and (84) into (72), we obtain in the same way

$$\zeta(|h_{K-1}|, -|h_K|) = \frac{\bigl(\tau - |h_{K-1}|\bigr)}{\tau^2} \int_{\mathbb{R}_+^3} \phi_{|h_{K-1}|,|h_K|}(\boldsymbol{\lambda}_{K-1:K+1}) \left(\frac{\left(r(\boldsymbol{\lambda}_{K-1:K+1})\right)^{1-\min(|h_{K-1}|,|h_K|)|} - 1}{1 - r(\boldsymbol{\lambda}_{K-1:K+1})} \right.$$
$$\left. + \max(\tau - |h_{K-1}| - |h_K| + 1, 0) \right) \mathrm{d}\boldsymbol{\lambda}_{K-1:K+1}. \tag{88}$$

Finally, by plugging (67), (68) and (87) into (64) for $k < K-1$, and equivalently (69), (70) and (88) into (64) for $k = K-1$, we finally obtain

$$[\boldsymbol{P}_{22}]_{k,k+1} = C_k = v(\tau, h_k, h_{k+1}) \frac{\beta^{\alpha_k + \alpha_{k+1} + \alpha_{k+2}}}{\Gamma(\alpha_k)\,\Gamma(\alpha_{k+1})\,\Gamma(\alpha_{k+2})} \int_{\mathbb{R}_+^3} \phi_{h_k,h_{k+1}}(\boldsymbol{z}) w(\boldsymbol{z}, \tau, h_k, h_{k+1}) \,\mathrm{d}\boldsymbol{z} \tag{89}$$

where functions $v(.)$ and $w(.)$ are defined in (22) and (23), respectively.

iii) Case D ($l = k$) and derivation of the diagonal terms of $\boldsymbol{P}_{22}$ and of $\boldsymbol{V}_{22}$: This case enables us to derive the diagonal terms of $\boldsymbol{P}_{22}$ (using the cases $h'_k = h_k$ and $h'_k = -h_k$), and the diagonal terms of $\boldsymbol{V}_{22}$ (in the case $h'_k = 0$), since

$$[\boldsymbol{P}_{22}]_{k,k} = \zeta(\boldsymbol{h}_k, \boldsymbol{h}_k) + \zeta(-\boldsymbol{h}_k, -\boldsymbol{h}_k) - \zeta(-\boldsymbol{h}_k, \boldsymbol{h}_k) - \zeta(\boldsymbol{h}_k, -\boldsymbol{h}_k) \tag{90}$$

and

$$[\boldsymbol{V}_{22}]_{k,k} = -h_k \zeta(\boldsymbol{h}_k, \boldsymbol{0}_{2K+1}). \tag{91}$$

The two first terms in (90) can be obtained by plugging (58) and (39) (with $\mathcal{J}_{h_k,h_k}$ given by (42)) into (36), i.e.,

$$\begin{aligned}
\zeta(\boldsymbol{h}_k, \boldsymbol{h}_k) = \zeta(-\boldsymbol{h}_k, -\boldsymbol{h}_k) &= \sum_{\boldsymbol{\ell} \in \mathbb{Z}^K} \int_{\mathbb{R}_+^{K+1}} \sum_{\boldsymbol{\kappa} \in \mathbb{N}^T} f\left(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_k\right) \mathrm{d}\boldsymbol{\lambda} \\
&= \sum_{\boldsymbol{\ell} \in \mathbb{Z}^K} \int_{\mathbb{R}_+^{K+1}} \Bigg(f(\boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_k) \underbrace{\sum_{\boldsymbol{\kappa} \in \mathbb{N}^T} f\left(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_k\right)}_{=1} \Bigg) \mathrm{d}\boldsymbol{\lambda} \\
&= \sum_{\boldsymbol{\ell} \in \mathbb{Z}^K} \int_{\mathbb{R}_+^{K+1}} \left(\prod_{i=1}^{K+1} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} \exp\left(-\beta \lambda_i\right) \right) \frac{1}{\tau^K} \mathbb{I}_{\mathcal{J}_{h_k,h_k}}(\boldsymbol{\ell})\, \mathrm{d}\boldsymbol{\lambda} \\
&= \left(\frac{1}{\tau^K} \sum_{\boldsymbol{\ell} \in \mathbb{Z}^K} \mathbb{I}_{\mathcal{J}_{h_k,h_k}}(\boldsymbol{\ell}) \right) \prod_{i=1}^{K+1} \underbrace{\left(\frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} \int_{\mathbb{R}_+} \lambda_i^{\alpha_i - 1} \exp\left(-\beta \lambda_i\right) \mathrm{d}\lambda_i \right)}_{=1} \\
&= \begin{cases} \frac{(\tau - |h_k|)^2}{\tau^2} & \text{if } k \le K-1 \text{ and } |h_k| \le \tau \\ \frac{\tau - |h_K|}{\tau} & \text{if } k = K \text{ and } |h_K| \le \tau \\ 0 & \text{if } |h_k| > \tau \end{cases} \\
&= u(\tau, h_k).
\end{aligned} \tag{92}$$

The two other terms in (90) can be obtained by plugging (59) and (39) (with $\mathcal{J}_{h_k,-h_k}$ given by (42)) into (36), i.e.,

$$\begin{aligned}
\zeta(\boldsymbol{h}_k, -\boldsymbol{h}_k) &= \sum_{\boldsymbol{\ell} \in \mathbb{Z}^K} \int_{\mathbb{R}_+^{K+1}} \left(\prod_{i=1}^{K+1} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} \exp\left(-\beta \lambda_i\right) \right) \frac{1}{\tau^K} \mathbb{I}_{\mathcal{J}_{h_k,-h_k}}(\boldsymbol{\ell})\, \rho^{2|h_k|}(\lambda_k, \lambda_{k+1})\, \mathrm{d}\boldsymbol{\lambda} \\
&= \left(\frac{1}{\tau^K} \sum_{\boldsymbol{\ell} \in \mathbb{Z}^K} \mathbb{I}_{\mathcal{J}_{h_k,-h_k}}(\boldsymbol{\ell}) \right) \frac{\beta^{\alpha_k + \alpha_{k+1}}}{\Gamma(\alpha_k)\Gamma(\alpha_{k+1})} \\
&\quad \times \left(\int_{\mathbb{R}_+^2} \lambda_k^{\alpha_k - 1} \lambda_{k+1}^{\alpha_{k+1} - 1} \exp\left\{ -\beta\left(\lambda_k + \lambda_{k+1}\right) - |h_k| \left(\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \right)^2 \right\} \mathrm{d}\lambda_k\, \mathrm{d}\lambda_{k+1} \right) \\
&= u(\tau, 2h_k)\Phi(2h_k) \qquad (93) \\
&= \zeta(-\boldsymbol{h}_k, \boldsymbol{h}_k), \qquad (94)
\end{aligned}$$

where $u(\tau, h_k)$ is defined in (21). Finally, by plugging (92) and (93) into (90), we obtain the diagonal terms of $[\boldsymbol{P}_{22}]_{k,k}$, for $k = 1, \ldots, K$

$$[\boldsymbol{P}_{22}]_{k,k} = B_k = 2\big(u(\tau, h_k) - u(\tau, 2h_k)\Phi(2h_k)\big). \tag{95}$$

Finally, with (63), (89) and (95), we retrieve the results given in (28), (29) and (30).

To conclude this section, the expression of $\zeta(\boldsymbol{h}_k, \boldsymbol{0}_{2K+1})$ can be obtained by plugging (39) (with $\mathcal{J}_{h_k,0}$ given by (42) with $h'_k = 0$) and (60) into (36). This yields

$$\begin{aligned}
\zeta(\boldsymbol{h}_k, \boldsymbol{0}_{2K+1}) &= \sum_{\boldsymbol{\ell} \in \mathbb{Z}^K} \int_{\mathbb{R}_+^{K+1}} \left(\prod_{i=1}^{K+1} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} \exp\left(-\beta \lambda_i\right) \right) \frac{1}{\tau^K} \mathbb{I}_{\mathcal{J}_{h_k,0}}(\boldsymbol{\ell})\, \rho^{|h_k|}(\lambda_k, \lambda_{k+1})\, \mathrm{d}\boldsymbol{\lambda} \\
&= \frac{1}{\tau^K} \left(\sum_{\boldsymbol{\ell} \in \mathbb{Z}^K} \mathbb{I}_{\mathcal{J}_{h_k,0}}(\boldsymbol{\ell}) \right) \frac{\beta^{\alpha_k + \alpha_{k+1}}}{\Gamma(\alpha_k)\Gamma(\alpha_{k+1})} \\
&\quad \times \left(\int_{\mathbb{R}_+^2} \lambda_k^{\alpha_k - 1} \lambda_{k+1}^{\alpha_{k+1} - 1} \exp\left\{ -\beta\left(\lambda_k + \lambda_{k+1}\right) - |h_k| \frac{\left(\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \right)^2}{2} \right\} \mathrm{d}\lambda_k\, \mathrm{d}\lambda_{k+1} \right) \\
&= u(\tau, h_k)\Phi(h_k)
\end{aligned} \tag{96}$$

which directly leads to the result we give in (24), i.e.,

$$[\boldsymbol{V}_{22}]_{k,k} = -h_k\, u\left(\tau, h_k\right) \Phi\left(h_k\right). \tag{97}$$

B. Derivation of $\boldsymbol{P}_{11}$

$$
\begin{aligned}
[\boldsymbol{P}_{11}]_{k,l} &= \mathbb{E}_{\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{t}}\left\{\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k}\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_l}\right\} \\
&= \int_{\mathbb{R}_+^{K+1}}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^K} f(\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})\Bigg[\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k}\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_l}f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}) \\
&\qquad + \frac{\partial \ln f(\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_l}\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k}f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}) \\
&\qquad + \frac{\partial \ln f(\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k}\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_l}f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}) \\
&\qquad + \frac{\partial \ln f(\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k}\frac{\partial \ln f(\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_l}\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})\Bigg]\,\mathrm{d}\boldsymbol{\lambda} \qquad (98)
\end{aligned}
$$

From (4)

$$
\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k} = \ell_{k-1}-\ell_k+\frac{1}{\lambda_k}\sum_{t=\ell_{k-1}+1}^{\ell_k}\kappa_t \qquad (99)
$$

and from (2) and (3)

$$
\frac{\partial \ln f(\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k} = (\alpha_k-1)\frac{1}{\lambda_k}-\beta. \qquad (100)
$$

It is straightforward that

$$
\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T} f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}) = 1. \qquad (101)
$$

On the other hand,

$$
\begin{aligned}
\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k}f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}) &= \sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k} \\
&= \frac{\partial}{\partial\lambda_k}\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}) \\
&= 0 \qquad (102)
\end{aligned}
$$

using (101).

$$
\begin{aligned}
&\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k}\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_l}f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}) \\
&\quad = (\ell_{k-1}-\ell_k)(\ell_{l-1}-\ell_l)+\frac{(\ell_{l-1}-\ell_l)}{\lambda_k}\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left[\left(\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right)\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right] \\
&\qquad + \frac{(\ell_{k-1}-\ell_k)}{\lambda_l}\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left[\left(\sum_{t'=\ell_{l-1}+1}^{\ell_l}\kappa_{t'}\right)\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right] \qquad (103) \\
&\qquad + \frac{1}{\lambda_k\lambda_l}\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left[\left(\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right)\left(\sum_{t''=\ell_{l-1}+1}^{\ell_l}\kappa_{t''}\right)\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]
\end{aligned}
$$

Let us first develop

$$
\begin{aligned}
\left(\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right)\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i) &= \left(\prod_{\substack{i=1\\i\neq k}}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right) \\
&\quad\times\sum_{t'=\ell_{k-1}+1}^{\ell_k}\left[\left(\prod_{\substack{t=\ell_{k-1}+1\\t\neq t'}}^{\ell_k}\frac{\lambda_k^{\kappa_t}}{\kappa_t!}\exp(-\lambda_k)\right)\frac{\lambda_k^{\kappa_{t'}}}{(\kappa_{t'}-1)!}\exp(-\lambda_k)\right] \qquad (104)
\end{aligned}
$$

Thus

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left[\left(\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right)\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]$$
$$=\sum_{[\boldsymbol{\kappa}_{1:\ell_{k-1}}^T,\boldsymbol{\kappa}_{\ell_k+1:T}^T]^T\in\mathbb{N}^{T-(\ell_k-\ell_{k-1})}}\left(\prod_{\substack{i=1\\ i\neq k}}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right)$$
$$\times\sum_{\boldsymbol{\kappa}_{\ell_{k-1}+1:\ell_k}\in\mathbb{N}^{\ell_k-\ell_{k-1}}}\left[\sum_{t'=\ell_{k-1}+1}^{\ell_k}\left[\left(\prod_{\substack{t=\ell_{k-1}+1\\ t\neq t'}}^{\ell_k}\frac{\lambda_k^{\kappa_t}}{\kappa_t!}\exp(-\lambda_k)\right)\frac{\lambda_k^{\kappa_{t'}}}{(\kappa_{t'}-1)!}\exp(-\lambda_k)\right]\right] \tag{105}$$

On the one hand

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\prod_{\substack{i=1\\ i\neq k}}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right)=\prod_{\substack{i=1\\ i\neq k}}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\left(\sum_{\kappa_t\in\mathbb{N}}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right)$$
$$=1^{T-(\ell_k-\ell_{k-1})}$$
$$=1. \tag{106}$$

On the other hand

$$\sum_{\boldsymbol{\kappa}_{\ell_{k-1}+1:\ell_k}\in\mathbb{N}^{\ell_k-\ell_{k-1}}}\left[\sum_{t'=\ell_{k-1}+1}^{\ell_k}\left[\left(\prod_{\substack{t=\ell_{k-1}+1\\ t\neq t'}}^{\ell_k}\frac{\lambda_k^{\kappa_t}}{\kappa_t!}\exp(-\lambda_k)\right)\frac{\lambda_k^{\kappa_{t'}}}{(\kappa_{t'}-1)!}\exp(-\lambda_k)\right]\right]$$
$$=\sum_{t'=\ell_{k-1}+1}^{\ell_k}\left[\underbrace{\left(\sum_{\kappa\in\mathbb{N}}\frac{\lambda_k^{\kappa}}{\kappa!}\exp(-\lambda_k)\right)^{\ell_k-\ell_{k-1}-1}}_{=1}\sum_{\kappa_{t'}=1}^{+\infty}\frac{\lambda_k^{\kappa_{t'}}}{(\kappa_{t'}-1)!}\exp(-\lambda_k)\right]$$
$$=\sum_{t'=\ell_{k-1}+1}^{\ell_k}\lambda_k$$
$$=\lambda_k(\ell_k-\ell_{k-1}). \tag{107}$$

Then, we obtain

$$\frac{(\ell_{l-1}-\ell_l)}{\lambda_k}\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left[\left(\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right)\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]=(\ell_{l-1}-\ell_l)(\ell_k-\ell_{k-1}) \tag{108}$$

and

$$\frac{(\ell_{k-1}-\ell_k)}{\lambda_l}\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left[\left(\sum_{t'=\ell_{l-1}+1}^{\ell_l}\kappa_{t'}\right)\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]=(\ell_{k-1}-\ell_k)(\ell_l-\ell_{l-1}). \tag{109}$$

The writing of the fourth term in the right hand side of (103) depends on whether $k=l$ or $k\neq l$.

- For $k\neq l$, using the same kind of manipulations as in equations (104) to (107), we find

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left[\left(\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right)\left(\sum_{t''=\ell_{l-1}+1}^{\ell_l}\kappa_{t''}\right)\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]=\left(\sum_{t'=\ell_{k-1}+1}^{\ell_k}\lambda_k\right)\left(\sum_{t''=\ell_{l-1}+1}^{\ell_l}\lambda_l\right)$$
$$=\lambda_k(\ell_k-\ell_{k-1})\lambda_l(\ell_l-\ell_{l-1}) \tag{110}$$

which finally leads, by plugging (108), (109) and (110) into (103), to

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\frac{\partial\ln f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k}\frac{\partial\ln f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_l}f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})=0. \tag{111}$$

• Conversely, if $k = l$, then (110) becomes

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left[\left(\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right)^2\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]$$
$$=\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left[\left(\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}^2+2\sum_{u=\ell_{k-1}+1}^{\ell_k}\sum_{v=u+1}^{\ell_k}\kappa_u\kappa_v\right)\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]. \quad (112)$$

On the one hand, we have

$$\begin{aligned}\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left[\left(\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}^2\right)\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right] &= \sum_{t'=\ell_{k-1}+1}^{\ell_k}\sum_{\kappa_{t'}=1}^{+\infty}\kappa_{t'}^2\frac{\lambda_k^{\kappa_{t'}}}{\kappa_{t'}!}\exp(-\lambda_k)\\
&= \sum_{t'=\ell_{k-1}+1}^{\ell_k}\left[\lambda_k\exp(-\lambda_k)\sum_{\kappa_{t'}=1}^{+\infty}\frac{\kappa_{t'}\lambda_k^{\kappa_{t'}-1}}{(\kappa_{t'}-1)!}\right]\\
&= \sum_{t'=\ell_{k-1}+1}^{\ell_k}\left[\lambda_k\exp(-\lambda_k)\sum_{\kappa_{t'}=1}^{+\infty}\frac{\mathrm{d}}{\mathrm{d}\lambda_k}\frac{\lambda_k^{\kappa_{t'}}}{(\kappa_{t'}-1)!}\right]\\
&= \sum_{t'=\ell_{k-1}+1}^{\ell_k}\left[\lambda_k\exp(-\lambda_k)\frac{\mathrm{d}}{\mathrm{d}\lambda_k}\left(\sum_{\kappa_{t'}=1}^{+\infty}\frac{\lambda_k^{\kappa_{t'}}}{(\kappa_{t'}-1)!}\right)\right]\\
&= \sum_{t'=\ell_{k-1}+1}^{\ell_k}\left[\lambda_k\exp(-\lambda_k)\frac{\mathrm{d}}{\mathrm{d}\lambda_k}\big(\lambda_k\exp(\lambda_k)\big)\right]\\
&= \sum_{t'=\ell_{k-1}+1}^{\ell_k}[\lambda_k(\lambda_k+1)]\\
&= (\ell_k-\ell_{k-1})\lambda_k(\lambda_k+1) \end{aligned} \quad (113)$$

and on the other hand, we have

$$\begin{aligned}&2\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left[\left(\sum_{u=\ell_{k-1}+1}^{\ell_k}\sum_{v=u+1}^{\ell_k}\kappa_u\kappa_v\right)\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]\\
&\quad= 2\sum_{u=\ell_{k-1}+1}^{\ell_k}\sum_{v=u+1}^{\ell_k}\left[\left(\sum_{\kappa_u=1}^{+\infty}\frac{\lambda_k^{\kappa_u}}{(\kappa_u-1)!}\exp(-\lambda_k)\right)\left(\sum_{\kappa_v=1}^{+\infty}\frac{\lambda_k^{\kappa_v}}{(\kappa_v-1)!}\exp(-\lambda_k)\right)\right]\\
&\quad= 2\sum_{u=\ell_{k-1}+1}^{\ell_k}\sum_{v=u+1}^{\ell_k}\lambda_k^2\\
&\quad= 2\lambda_k^2\left(\ell_k(\ell_k-\ell_{k-1})-\frac{(\ell_k-\ell_{k-1})(\ell_k+\ell_{k-1}+1)}{2}\right)\\
&\quad= \lambda_k^2(\ell_k-\ell_{k-1})(\ell_k-\ell_{k-1}-1). \end{aligned} \quad (114)$$

Thus, if $k = l$, gathering (112), (113) and (114) together and after plugging the result into (103), we finally obtain

$$\begin{aligned}&\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\frac{\partial\ln f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k}\frac{\partial\ln f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_l}f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})\\
&\quad= -(\ell_k-\ell_{k-1})^2+\frac{1}{\lambda_k^2}\left[\lambda_k(\lambda_k+1)(\ell_k-\ell_{k-1})+\lambda_k^2(\ell_k-\ell_{k-1})(\ell_k-\ell_{k-1}-1)\right]\\
&\quad= \frac{\ell_k-\ell_{k-1}}{\lambda_k}. \end{aligned} \quad (115)$$

Finally, considering (100), (101), (102), (111) and (115), and after plugging them into (98), we obtain

$$\begin{aligned}[\boldsymbol{P}_{11}]_{k,l} = &\int_{\mathbb{R}_+^{K+1}} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^T} \left[\frac{\ell_k-\ell_{k-1}}{\lambda_k}\delta_{k,l}\cdot\frac{1}{\tau^K}\mathbb{I}_{\mathcal{J}_{0,0}}(\boldsymbol{\ell})\cdot\prod_{i=1}^{K+1}\frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)}\lambda_i^{\alpha_i-1}\exp(-\beta\lambda_i)\right]\mathrm{d}\boldsymbol{\lambda} \\ &+\int_{\mathbb{R}_+^{K+1}} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^T}\left[\left(\frac{\alpha_k-1}{\lambda_k}-\beta\right)\left(\frac{\alpha_l-1}{\lambda_l}-\beta\right)\cdot\frac{1}{\tau^K}\mathbb{I}_{\mathcal{J}_{0,0}}(\boldsymbol{\ell})\cdot\prod_{i=1}^{K+1}\frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)}\lambda_i^{\alpha_i-1}\exp(-\beta\lambda_i)\right]\mathrm{d}\boldsymbol{\lambda}.\end{aligned} \tag{116}$$

The first term in the right-hand side of (116) can be developed as

$$\begin{aligned}&\int_{\mathbb{R}_+^{K+1}} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^T} \left[\frac{\ell_k-\ell_{k-1}}{\lambda_k}\delta_{k,l}\cdot\frac{1}{\tau^K}\mathbb{I}_{\mathcal{J}_{0,0}}(\boldsymbol{\ell})\cdot\prod_{i=1}^{K+1}\frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)}\lambda_i^{\alpha_i-1}\exp(-\beta\lambda_i)\right]\mathrm{d}\boldsymbol{\lambda} \\
&= \frac{\delta_{k,l}}{\tau^K}\int_{\mathbb{R}_+^{K+1}}\left[\left(\sum_{\boldsymbol{\ell}\in\mathcal{J}_{0,0}}\frac{\ell_k-\ell_{k-1}}{\lambda_k}\right)\prod_{i=1}^{K+1}\frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)}\lambda_i^{\alpha_i-1}\exp(-\beta\lambda_i)\right]\mathrm{d}\boldsymbol{\lambda} \\
&= \frac{\delta_{k,l}}{\tau^k}\int_{\mathbb{R}_+^{K+1}}\left[\left(\frac{1}{\lambda_k}\sum_{\boldsymbol{\ell}_{1:k-2}\in\mathcal{J}_{1:k-2,0,0}}\sum_{\ell_{k-1}=\ell_{k-2}+1}^{\ell_{k-2}+\tau}\sum_{\ell_k=\ell_{k-1}+1}^{\ell_{k-1}+\tau}(\ell_k-\ell_{k-1})\right)\prod_{i=1}^{K+1}\frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)}\lambda_i^{\alpha_i-1}\exp(-\beta\lambda_i)\right]\mathrm{d}\boldsymbol{\lambda} \\
&= \frac{\delta_{k,l}}{\tau^k}\int_{\mathbb{R}_+^{K+1}}\left[\left(\frac{1}{\lambda_k}\sum_{\boldsymbol{\ell}_{1:k-2}\in\mathcal{J}_{1:k-2,0,0}}\sum_{\ell_{k-1}=\ell_{k-2}+1}^{\ell_{k-2}+\tau}\left(\frac{\tau(2\ell_{k-1}+\tau+1)}{2}-\tau\ell_{k-1}\right)\right)\prod_{i=1}^{K+1}\frac{\beta^{\alpha_i}}{\Gamma(\alpha_k)}\lambda_i^{\alpha_i-1}\exp(-\beta\lambda_i)\right]\mathrm{d}\boldsymbol{\lambda} \\
&= \delta_{k,l}\frac{\tau+1}{2}\int_{\mathbb{R}_+^{K+1}}\left(\prod_{\substack{i=1\\i\neq k}}^{K+1}\frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)}\lambda_i^{\alpha_i-1}\exp(-\beta\lambda_i)\right)\frac{\beta^{\alpha_k}}{\Gamma(\alpha_k)}\lambda_k^{\alpha_k-2}\exp(-\beta\lambda_k)\,\mathrm{d}\boldsymbol{\lambda} \\
&= \delta_{k,l}\frac{\tau+1}{2}\frac{\beta^{\alpha_k}}{\Gamma(\alpha_k)}\int_{\mathbb{R}_+^{K+1}}\lambda_k^{\alpha_k-2}\exp(-\beta\lambda_k)\,\mathrm{d}\lambda_k \\
&= \delta_{k,l}\frac{\tau+1}{2}\frac{\beta^{\alpha_k}}{\Gamma(\alpha_k)}\frac{\Gamma(\alpha_k-1)}{\beta^{\alpha_k-1}} \\
&= \delta_{k,l}\frac{\beta(\tau+1)}{2(\alpha_k-1)}\end{aligned} \tag{117}$$

in which $\delta_{k,l}$ denotes the Dirac delta. The second term in the right-hand side of (116) can be developed, for $k\neq l$, as

$$\begin{aligned}&\int_{\mathbb{R}_+^{K+1}} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^T}\left[\left(\frac{\alpha_k-1}{\lambda_k}-\beta\right)\left(\frac{\alpha_l-1}{\lambda_l}-\beta\right)\cdot\frac{1}{\tau^K}\mathbb{I}_{\mathcal{J}_{0,0}}(\boldsymbol{\ell})\cdot\prod_{i=1}^{K+1}\frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)}\lambda_i^{\alpha_i-1}\exp(-\beta\lambda_i)\right]\mathrm{d}\boldsymbol{\lambda} \\
&\qquad= \underbrace{\frac{1}{\tau^K}\left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}^T}\mathbb{I}_{\mathcal{J}_{0,0}}(\boldsymbol{\ell})\right)}_{=1}\int_{\mathbb{R}_+^{K+1}}\left[\left(\frac{\alpha_k-1}{\lambda_k}-\beta\right)\left(\frac{\alpha_l-1}{\lambda_l}-\beta\right)\prod_{i=1}^{K+1}\frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)}\lambda_i^{\alpha_i-1}\exp(-\beta\lambda_i)\right]\mathrm{d}\boldsymbol{\lambda} \\
&\qquad= \prod_{\substack{i=1\\i\neq k,i\neq l}}^{K+1}\underbrace{\left(\frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)}\int_{\mathbb{R}_+}\lambda_i^{\alpha_i-1}\exp(-\beta\lambda_i)\,\mathrm{d}\lambda_i\right)}_{=1}\frac{\beta^{\alpha_k}}{\Gamma(\alpha_k)}\int_{\mathbb{R}_+}\left(\frac{\alpha_k-1}{\lambda_k}-\beta\right)\lambda_k^{\alpha_k-1}\exp(-\beta\lambda_k)\,\mathrm{d}\lambda_k \\
&\qquad\qquad\times\frac{\beta^{\alpha_l}}{\Gamma(\alpha_l)}\int_{\mathbb{R}_+}\left(\frac{\alpha_l-1}{\lambda_l}-\beta\right)\lambda_l^{\alpha_l-1}\exp(-\beta\lambda_l)\,\mathrm{d}\lambda_l \\
&\qquad= \frac{\beta^{\alpha_k}}{\Gamma(\alpha_k)}\frac{\beta^{\alpha_l}}{\Gamma(\alpha_l)}\left[(\alpha_k-1)\left(\int_{\mathbb{R}_+}\lambda_k^{\alpha_k-2}\exp(-\beta\lambda_k)\,\mathrm{d}\lambda_k\right)-\beta\frac{\Gamma(\alpha_k)}{\beta^{\alpha_k}}\right] \\
&\qquad\qquad\times\left[(\alpha_l-1)\left(\int_{\mathbb{R}_+}\lambda_l^{\alpha_l-2}\exp(-\beta\lambda_l)\,\mathrm{d}\lambda_l\right)-\beta\frac{\Gamma(\alpha_l)}{\beta^{\alpha_l}}\right] \\
&\qquad= \frac{\beta^{\alpha_k}}{\Gamma(\alpha_k)}\frac{\beta^{\alpha_l}}{\Gamma(\alpha_l)}\left[(\alpha_k-1)\frac{\Gamma(\alpha_k-1)}{\beta^{\alpha_k-1}}-\frac{\Gamma(\alpha_k)}{\beta^{\alpha_k-1}}\right]\cdot\left[(\alpha_l-1)\frac{\Gamma(\alpha_l-1)}{\beta^{\alpha_l-1}}-\frac{\Gamma(\alpha_l)}{\beta^{\alpha_l-1}}\right] \\
&\qquad= 0,\end{aligned} \tag{118}$$

provided that $\alpha_k, \alpha_l > 2$. Conversely, if $k = l$, we obtain

$$\int_{\mathbb{R}_+^{K+1}} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^T} \left[\left(\frac{\alpha_k - 1}{\lambda_k} - \beta \right)^2 \cdot \frac{1}{\tau^K} \mathbb{I}_{\mathcal{J}_{0,0}}(\boldsymbol{\ell}) \cdot \prod_{i=1}^{K+1} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} \exp(-\beta\lambda_i) \right] \mathrm{d}\boldsymbol{\lambda}$$
$$= \frac{\beta^{\alpha_k}}{\Gamma(\alpha_k)} \int_{\mathbb{R}_+} \left(\frac{\alpha_k - 1}{\lambda_k} - \beta \right)^2 \lambda_k^{\alpha_k - 1} \exp(-\beta\lambda_k) \, \mathrm{d}\lambda_k$$
$$= \frac{\beta^{\alpha_k}}{\Gamma(\alpha_k)} \left[(\alpha_k - 1)^2 \left(\int_{\mathbb{R}_+} \lambda_k^{\alpha_k - 3} \exp(-\beta\lambda_k) \, \mathrm{d}\lambda_k \right) - 2\beta(\alpha_k - 1) \left(\int_{\mathbb{R}_+} \lambda_k^{\alpha_k - 2} \exp(-\beta\lambda_k) \, \mathrm{d}\lambda_k \right) + \beta^2 \frac{\Gamma(\alpha_k)}{\beta^{\alpha_k}} \right]$$
$$= \frac{\beta^{\alpha_k}}{\Gamma(\alpha_k)} \left[(\alpha_k - 1)^2 \frac{\Gamma(\alpha_k - 2)}{\beta^{\alpha_k - 2}} - 2(\alpha_k - 1) \frac{\Gamma(\alpha_k - 1)}{\beta^{\alpha_k - 2}} + \frac{\Gamma(\alpha_k)}{\beta^{\alpha_k - 2}} \right]$$
$$= \frac{\beta^{\alpha_k}}{\Gamma(\alpha_k)} \frac{(\alpha_k - 1)^2 \, \Gamma(\alpha_k - 2) - \Gamma(\alpha_k)}{\beta^{\alpha_k - 2}}$$
$$= \frac{\beta^2 (\alpha_k - 1) \, \Gamma(\alpha_k - 2)}{(\alpha_k - 1)(\alpha_k - 2)\Gamma(\alpha_k - 2)}$$
$$= \frac{\beta^2}{\alpha_k - 2}, \tag{119}$$

provided that $\alpha_k > 2$.

Finally, considering (117), (118) and (119), and after plugging them into (116), we retrieve (25), i.e.,

$$[\boldsymbol{P}_{11}]_{k,l} = \left(\frac{\beta\,(\tau + 1)}{2\,(\alpha_k - 1)} + \frac{\beta^2}{\alpha_k - 2} \right) \delta_{k,l}. \tag{120}$$

C. Derivation of $\boldsymbol{P}_{12}$

$$[\boldsymbol{P}_{12}]_{k,l} = \mathbb{E}_{\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{t}} \left\{ \frac{\partial \ln f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})}{\partial \lambda_k} \left(\sqrt{\frac{f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_l)}{f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})}} - \sqrt{\frac{f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} - \boldsymbol{h}_l)}{f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})}} \right) \right\} \tag{121}$$

Let us first derive the quantity

$$\mathbb{E}_{\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{t}} \left\{ \frac{\partial \ln f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})}{\partial \lambda_k} \sqrt{\frac{f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_l)}{f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})}} \right\}$$
$$= \int_{\mathbb{R}_+^{K+1}} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^K} \sum_{\boldsymbol{\kappa}\in\mathbb{N}^T} \frac{\partial \ln f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})}{\partial \lambda_k} \sqrt{f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_l) f(\boldsymbol{x} = \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})} \, \mathrm{d}\boldsymbol{\lambda}$$
$$= \int_{\mathbb{R}_+^{K+1}} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^K} \Bigg[\sqrt{f(\boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_l) f(\boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})}$$
$$\times \Bigg(\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T} \left(\frac{\partial \ln f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})}{\partial \lambda_k} \sqrt{f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_l) f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})} \right)$$
$$+ \frac{\partial \ln f(\boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})}{\partial \lambda_k} \sum_{\boldsymbol{\kappa}\in\mathbb{N}^T} \sqrt{f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_l) f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})} \Bigg) \Bigg] \, \mathrm{d}\boldsymbol{\lambda}$$
$$= \int_{\mathbb{R}_+^{K+1}} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^K} \Bigg[\sqrt{f(\boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_l) f(\boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})}$$
$$\times \Bigg(\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T} \left(\left[\ell_{k-1} - \ell_k + \frac{1}{\lambda_k} \sum_{t'=\ell_{k-1}+1}^{\ell_k} \kappa_{t'} \right] \sqrt{f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_l) f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})} \right)$$
$$+ \left[\frac{\alpha_k - 1}{\lambda_k} - \beta \right] \sum_{\boldsymbol{\kappa}\in\mathbb{N}^T} \sqrt{f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_l) f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})} \Bigg) \Bigg] \, \mathrm{d}\boldsymbol{\lambda}. \tag{122}$$

Note that the quantity $\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T} \sqrt{f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell} + \boldsymbol{h}_l) f(\boldsymbol{x} = \boldsymbol{\kappa} | \boldsymbol{\lambda}, \boldsymbol{t} = \boldsymbol{\ell})}$ is nothing else than $\acute{\zeta}(\boldsymbol{\lambda}, \boldsymbol{\ell}, \boldsymbol{h}_l, \boldsymbol{0}_{2K+1})$, which is given in (60).

In (122), we develop

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\ell_{k-1}-\ell_k+\frac{1}{\lambda_k}\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right]\sqrt{f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_l)f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}\right)$$
$$=(\ell_{k-1}-\ell_k)\,\rho^{|h_l|}(\lambda_l,\lambda_{l+1})+\frac{1}{\lambda_k}\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right]\left[\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+\delta_{i-1,l}\max(h_l,0)+1}^{\ell_i-\delta_{i,l}\max(-h_l,0)}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]\right.$$
$$\left.\times\prod_{t=\ell_l-\max(-h_l,0)+1}^{\ell_l+\max(h_l,0)}\frac{\sqrt{(\lambda_l\lambda_{l+1})^{\kappa_t}}}{\kappa_t!}\exp\left\{-\frac{\lambda_l+\lambda_{l+1}}{2}\right\}\right). \tag{123}$$

The derivation of the second term in the right hand side of (123) again depends upon the cases:

1) $l\neq k$ and $l\neq k-1$, case referred to as “ULT” (for “upper and lower triangles”);
2) $l=k$, case referred to as “D1” (for “1st diagonal”);
3) $l=k-1$, case referred to as “D2” (for “2nd diagonal”).

Details for each case are given in the following sections.

1) Case ULT ($l\neq k$ and $l\neq k-1$): We develop the second term in the right hand side of (123):

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right]\left[\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+\delta_{i-1,l}\max(h_l,0)+1}^{\ell_i-\delta_{i,l}\max(-h_l,0)}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]\prod_{t=\ell_l-\max(-h_l,0)+1}^{\ell_l+\max(h_l,0)}\frac{\sqrt{(\lambda_l\lambda_{l+1})^{\kappa_t}}}{\kappa_t!}\exp\left\{-\frac{\lambda_l+\lambda_{l+1}}{2}\right\}\right)$$
$$=\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\prod_{\substack{i=1\\i\neq k}}^{K+1}\prod_{t=\ell_{i-1}+\delta_{i-1,l}\max(h_l,0)+1}^{\ell_i-\delta_{i,l}\max(-h_l,0)}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]\sum_{t'=\ell_{k-1}+1}^{\ell_k}\left[\left(\prod_{\substack{t=\ell_{k-1}+1\\t\neq t'}}^{\ell_k}\frac{\lambda_k^{\kappa_t}}{\kappa_t!}\exp(-\lambda_k)\right)\frac{\lambda_k^{\kappa_{t'}}}{(\kappa_{t'}-1)!}\exp(-\lambda_k)\right]\right.$$
$$\left.\times\prod_{t=\ell_l-\max(-h_l,0)+1}^{\ell_l+\max(h_l,0)}\frac{\sqrt{(\lambda_l\lambda_{l+1})^{\kappa_t}}}{\kappa_t!}\exp\left\{-\frac{\lambda_l+\lambda_{l+1}}{2}\right\}\right)$$
$$=\left(\sum_{t'=\ell_{k-1}+1}^{\ell_k}\sum_{\kappa_{t'}=1}^{+\infty}\frac{\lambda_k^{\kappa_{t'}}}{(\kappa_{t'}-1)!}\exp(-\lambda_k)\right)\left(\sum_{\kappa=0}^{+\infty}\frac{\sqrt{(\lambda_l\lambda_{l+1})^{\kappa_t}}}{\kappa_t!}\exp\left\{-\frac{\lambda_l+\lambda_{l+1}}{2}\right\}\right)^{|h_l|}$$
$$=\lambda_k\,(\ell_k-\ell_{k-1})\,\rho^{|h_l|}(\lambda_l,\lambda_{l+1}) \tag{124}$$

Then, plugging (124) into (123), we simply find

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\ell_{k-1}-\ell_k+\frac{1}{\lambda_k}\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right]\sqrt{f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_l)f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}\right)=0 \tag{125}$$

for $k\neq l$, $k\neq l+1$.

2) Case D1 ($l=k$): The writing of the second term in the right hand side of (123) depends upon whether $h_k\gtrless 0$.

• Let us first assume that $h_k>0$. In this case, the second term in the right hand side of (123) becomes

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right]\left[\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+\delta_{i-1,k}h_k+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]\prod_{t=\ell_k+1}^{\ell_k+h_k}\frac{\sqrt{(\lambda_k\lambda_{k+1})^{\kappa_t}}}{\kappa_t!}\exp\left\{-\frac{\lambda_k+\lambda_{k+1}}{2}\right\}\right)$$
$$=\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\prod_{\substack{i=1\\i\neq k,k+1}}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]\sum_{t'=\ell_{k-1}+1}^{\ell_k}\left[\left(\prod_{\substack{t=\ell_{k-1}+1\\t\neq t'}}^{\ell_k}\frac{\lambda_k^{\kappa_t}}{\kappa_t!}\exp(-\lambda_k)\right)\frac{\lambda_k^{\kappa_{t'}}}{(\kappa_{t'}-1)!}\exp(-\lambda_k)\right]\right.$$
$$\left.\times\prod_{t=\ell_k+1}^{\ell_k+h_k}\frac{\sqrt{(\lambda_k\lambda_{k+1})^{\kappa_t}}}{\kappa_t!}\exp\left\{-\frac{\lambda_k+\lambda_{k+1}}{2}\right\}\prod_{t=\ell_k+h_k+1}^{\ell_{k+1}}\frac{\lambda_{k+1}^{\kappa_t}}{\kappa_t!}\exp(-\lambda_{k+1})\right)$$
$$=\lambda_k\,(\ell_k-\ell_{k-1})\,\rho^{|h_k|}(\lambda_k,\lambda_{k+1}) \tag{126}$$

in the same way as in the case ULT. We then have, again, for $h_k>0$

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\ell_{k-1}-\ell_k+\frac{1}{\lambda_k}\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right]\sqrt{f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_k)f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}\right)=0. \tag{127}$$

• In the converse case $h_k < 0$, (126) becomes

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right]\left[\prod_{i=1}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i+\delta_{i,k}h_k}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]\prod_{t=\ell_k+h_k+1}^{\ell_k}\frac{\sqrt{(\lambda_k\lambda_{k+1})^{\kappa_t}}}{\kappa_t!}\exp\Big\{-\frac{\lambda_k+\lambda_{k+1}}{2}\Big\}\right)$$

$$= \sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\sum_{t'=\ell_{k-1}+1}^{\ell_k+h_k}\kappa_{t'}+\sum_{t'=\ell_k+h_k+1}^{\ell_k}\kappa_{t'}\right]\left[\prod_{\substack{i=1\\i\neq k}}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]\right.$$

$$\left.\times\prod_{t=\ell_{k-1}+1}^{\ell_k+h_k}\frac{\lambda_k^{\kappa_t}}{\kappa_t!}\exp(-\lambda_k)\prod_{t=\ell_k+h_k+1}^{\ell_k}\frac{\sqrt{(\lambda_k\lambda_{k+1})^{\kappa_t}}}{\kappa_t!}\exp\Big\{-\frac{\lambda_k+\lambda_{k+1}}{2}\Big\}\right)$$

$$= \sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\prod_{\substack{i=1\\i\neq k}}^{K+1}\prod_{t=\ell_{i-1}+1}^{\ell_i}\frac{\lambda_i^{\kappa_t}}{\kappa_t!}\exp(-\lambda_i)\right]\left\{\left[\prod_{t=\ell_k+h_k+1}^{\ell_k}\frac{\sqrt{(\lambda_k\lambda_{k+1})^{\kappa_t}}}{\kappa_t!}\exp\Big\{-\frac{\lambda_k+\lambda_{k+1}}{2}\Big\}\right.\right.\right.$$

$$\left.\times\sum_{t'=\ell_{k-1}+1}^{\ell_k+h_k}\left[\left(\prod_{\substack{t=\ell_{k-1}+1\\t\neq t'}}^{\ell_k+h_k}\frac{\lambda_k^{\kappa_t}}{\kappa_t!}\exp(-\lambda_k)\right)\frac{\lambda_k^{\kappa_{t'}}}{(\kappa_{t'}-1)!}\exp(-\lambda_k)\right]\right]$$

$$+\left[\prod_{t=\ell_{k-1}+1}^{\ell_k+h_k}\frac{\lambda_k^{\kappa_t}}{\kappa_t!}\exp(-\lambda_k)\right.$$

$$\times\sum_{t'=\ell_k+h_k+1}^{\ell_k}\left[\left(\prod_{\substack{t=\ell_k+h_k+1\\t\neq t'}}^{\ell_k}\frac{\sqrt{(\lambda_k\lambda_{k+1})^{\kappa_t}}}{\kappa_t!}\exp\Big\{-\frac{\lambda_k+\lambda_{k+1}}{2}\Big\}\right)\right.$$

$$\left.\left.\left.\left.\times\frac{\sqrt{(\lambda_k\lambda_{k+1})^{\kappa_{t'}}}}{(\kappa_{t'}-1)!}\exp\Big\{-\frac{\lambda_k+\lambda_{k+1}}{2}\Big\}\right]\right]\right\}\right)$$

$$= \left(\sum_{\kappa=0}^{+\infty}\frac{\sqrt{(\lambda_k\lambda_{k+1})^{\kappa_t}}}{\kappa_t!}\exp\Big\{-\frac{\lambda_k+\lambda_{k+1}}{2}\Big\}\right)^{-h_k}\left[\sum_{t'=\ell_{k-1}+1}^{\ell_k+h_k}\left(\sum_{\kappa_{t'}=1}^{+\infty}\frac{\lambda_k^{\kappa_{t'}}}{(\kappa_{t'}-1)!}\exp(-\lambda_k)\right)\right]$$

$$+\sum_{t'=\ell_k+h_k+1}^{\ell_k}\left[\left(\sum_{\kappa=0}^{+\infty}\frac{\sqrt{(\lambda_k\lambda_{k+1})^{\kappa}}}{\kappa!}\exp\Big\{-\frac{\lambda_k+\lambda_{k+1}}{2}\Big\}\right)^{-h_k-1}\left(\sum_{\kappa_{t'}=1}^{+\infty}\frac{\sqrt{(\lambda_k\lambda_{k+1})^{\kappa_{t'}}}}{(\kappa_{t'}-1)!}\exp\Big\{-\frac{\lambda_k+\lambda_{k+1}}{2}\Big\}\right)\right]$$

$$= \rho^{-h_k}(\lambda_k,\lambda_{k+1})\,\lambda_k(\ell_k+h_k-\ell_{k-1})+\sum_{t'=\ell_k+h_k+1}^{\ell_k}\Big[\rho^{-h_k-1}(\lambda_k,\lambda_{k+1})\,\sqrt{\lambda_k\lambda_{k+1}}\,\rho(\lambda_k,\lambda_{k+1})\Big]$$

$$= \rho^{-h_k}(\lambda_k,\lambda_{k+1})\Big[\lambda_k(\ell_k-\ell_{k-1})+h_k\big(\lambda_k-\sqrt{\lambda_k\lambda_{k+1}}\big)\Big]$$

$$= \lambda_k\,\rho^{-h_k}(\lambda_k,\lambda_{k+1})\left[(\ell_k-\ell_{k-1})+h_k\left(1-\sqrt{\frac{\lambda_{k+1}}{\lambda_k}}\right)\right]. \tag{128}$$

Then, by plugging (128) into (123), we obtain

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\ell_{k-1}-\ell_k+\frac{1}{\lambda_k}\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right]\sqrt{f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_k)f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}\right)$$

$$= (-h_k)\left(\sqrt{\frac{\lambda_{k+1}}{\lambda_k}}-1\right)\rho^{-h_k}(\lambda_k,\lambda_{k+1}). \tag{129}$$

3) Case D2 ($l = k-1$): In this case as well, we have to distinguish between the two cases $h_{k-1} \gtrless 0$.

• Let us first assume that $h_{k-1} < 0$. This case can be handled by using exactly the same methodology as in the case D1 with $h_k > 0$ (see (126)). We then find, for $h_{k-1} < 0$

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\ell_{k-1}-\ell_k+\frac{1}{\lambda_k}\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right]\sqrt{f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_{k-1})f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}\right)=0. \tag{130}$$

• The converse case $h_{k-1} > 0$ can be handled also by using the exact same methodology as that used in the case D1 with

$h_k < 0$ (see (128)), so that we obtain

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\ell_{k-1}-\ell_k+\frac{1}{\lambda_k}\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right]\sqrt{f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_{k-1})f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}\right)$$
$$= h_{k-1}\left(\sqrt{\frac{\lambda_{k-1}}{\lambda_k}}-1\right)\rho^{h_{k-1}}(\lambda_{k-1},\lambda_k). \quad (131)$$

Finally, we can merge both cases D1 and D2 (i.e., whether $l=k$ or $l=k-1$) by writing

$$\sum_{\boldsymbol{\kappa}\in\mathbb{N}^T}\left(\left[\ell_{k-1}-\ell_k+\frac{1}{\lambda_k}\sum_{t'=\ell_{k-1}+1}^{\ell_k}\kappa_{t'}\right]\sqrt{f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_l)f(\boldsymbol{x}=\boldsymbol{\kappa}|\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}\right)$$
$$= \max(\pm h_l,0)\left(\left(\frac{\lambda_l}{\lambda_{l+1}}\right)^{\pm 1/2}-1\right)\rho^{|h_l|}(\lambda_l,\lambda_{l+1}) \quad (132)$$

where the "$\pm$" signs both are "$+$" signs if $l=k-1$, and they are "$-$" signs if $l=k$.

Completion of the derivation (122) and final expression of $\boldsymbol{P}_{12}$: From the foregoing, we can now complete the derivation of (122). Let us do it separately, according to the cases ULT, and D1-2.

i) Case ULT ($l\neq k-1$ and $l\neq k$): Carrying on the derivation started in (122), and using (125), we have

$$\mathbb{E}_{\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{t}}\left\{\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k}\sqrt{\frac{f(\boldsymbol{x}=\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_l)}{f(\boldsymbol{x}=\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}}\right\}$$
$$\begin{aligned}
&= \int_{\mathbb{R}_+^{K+1}}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^K}\left[\sqrt{f(\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_l)f(\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}\left(\frac{\alpha_k-1}{\lambda_k}-\beta\right)\rho^{|h_l|}(\lambda_l,\lambda_{l+1})\right]\mathrm{d}\boldsymbol{\lambda}\\
&= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^K}\int_{\mathbb{R}_+^{K+1}}\left[\left(\prod_{i=1}^{K+1}\frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)}\lambda_i^{\alpha_i-1}\exp(-\beta\lambda_i)\right)\frac{1}{\tau^K}\mathbb{I}_{\mathcal{J}_{h_l,0}}(\boldsymbol{\ell})\right.\\
&\qquad\left.\times\left(\frac{\alpha_k-1}{\lambda_k}-\beta\right)\exp\left\{-|h_l|\frac{(\sqrt{\lambda_{l+1}}-\sqrt{\lambda_l})^2}{2}\right\}\right]\mathrm{d}\boldsymbol{\lambda}\\
&= \left(\frac{1}{\tau^K}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^K}\mathbb{I}_{\mathcal{J}_{h_l,0}}(\boldsymbol{\ell})\right)\left[\prod_{\substack{i=1\\ i\neq k,l,l+1}}^{K+1}\int_{\mathbb{R}_+^{K-1}}\frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)}\lambda_i^{\alpha_i-1}\exp(-\beta\lambda_i)\,\mathrm{d}\lambda_i\right]\Phi(h_l)\\
&\qquad\times\frac{\beta^{\alpha_k}}{\Gamma(\alpha_k)}\underbrace{\int_{\mathbb{R}_+}\left(\frac{\alpha_k-1}{\lambda_k}-\beta\right)\lambda_k^{\alpha_k-1}\exp(-\beta\lambda_k)\,\mathrm{d}\lambda_k}_{=0}\\
&= 0
\end{aligned} \quad (133)$$

by using the same arguments as in (118), for example. Of course, in (133), when replacing h_l with $-h_l$, we obtain zero as well. Then, by plugging (133) into (121) twice (once with $+\boldsymbol{h}_l$ and once with $-\boldsymbol{h}_l$), we obtain, for $l\neq k-1$ and $l\neq k$

$$[\boldsymbol{P}_{12}]_{k,l}=0. \quad (134)$$

ii) Cases D1 and D2 ($l=k-1$ or $l=k$): By plugging (132) into (122), we obtain

$$\mathbb{E}_{\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{t}}\left\{\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial\lambda_k}\sqrt{\frac{f(\boldsymbol{x}=\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_l)}{f(\boldsymbol{x}=\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}}\right\}$$
$$\begin{aligned}
&= \int_{\mathbb{R}_+^{K+1}}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^K}\left[\sqrt{f(\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}+\boldsymbol{h}_l)f(\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}\left(\frac{\alpha_k-1}{\lambda_k}-\beta+\max(\pm h_l,0)\left(\left(\frac{\lambda_l}{\lambda_{l+1}}\right)^{\pm1/2}-1\right)\right)\rho^{|h_l|}(\lambda_l,\lambda_{l+1})\right]\mathrm{d}\boldsymbol{\lambda}\\
&= \left(\frac{1}{\tau^K}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^K}\mathbb{I}_{\mathcal{J}_{h_l,0}}(\boldsymbol{\ell})\right)\left[\prod_{\substack{i=1\\ i\neq l,l+1}}^{K+1}\int_{\mathbb{R}_+^{K-1}}\frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)}\lambda_i^{\alpha_i-1}\exp(-\beta\lambda_i)\,\mathrm{d}\lambda_i\right]\cdot\left[\frac{\beta^{\alpha_l+\alpha_{l+1}}}{\Gamma(\alpha_l)\Gamma(\alpha_{l+1})}\right.\\
&\qquad\left.\times\int_{\mathbb{R}_+^2}\left(\frac{\alpha_k-1}{\lambda_k}-\beta+\max(\pm h_l,0)\left(\left(\frac{\lambda_l}{\lambda_{l+1}}\right)^{\pm1/2}-1\right)\right)\exp\left\{-\beta(\lambda_l+\lambda_{l+1})-|h_l|\frac{\left(\sqrt{\lambda_{l+1}}-\sqrt{\lambda_l}\right)^2}{2}\right\}\mathrm{d}\boldsymbol{\lambda}_{l:l+1}\right]\\
&= u(\tau,h_l)\frac{\beta^{\alpha_l+\alpha_{l+1}}}{\Gamma(\alpha_l)\Gamma(\alpha_{l+1})}\int_{\mathbb{R}_+^2}\left(\frac{\alpha_k-1}{\lambda_k}-\beta+\max(\pm h_l,0)\left(\left(\frac{\lambda_l}{\lambda_{l+1}}\right)^{\pm1/2}-1\right)\right)\varphi_{h_l}(\boldsymbol{\lambda}_{l:l+1})\,\mathrm{d}\boldsymbol{\lambda}_{l:l+1}.
\end{aligned} \quad (135)$$

When replacing h_l with $-h_l$ in (135), we obtain, accordingly

$$\mathbb{E}_{\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{t}}\left\{\frac{\partial \ln f(\boldsymbol{x}=\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}{\partial \lambda_k}\sqrt{\frac{f(\boldsymbol{x}=\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell}-\boldsymbol{h}_l)}{f(\boldsymbol{x}=\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{t}=\boldsymbol{\ell})}}\right\}$$
$$= u(\tau,h_l)\,\frac{\beta^{\alpha_l+\alpha_{l+1}}}{\Gamma(\alpha_l)\,\Gamma(\alpha_{l+1})}\int_{\mathbb{R}_+^2}\left(\frac{\alpha_k-1}{\lambda_k}-\beta+\max(\mp h_l,0)\left(\left(\frac{\lambda_l}{\lambda_{l+1}}\right)^{\pm 1/2}-1\right)\right)\varphi_{h_l}(\boldsymbol{\lambda}_{l:l+1})\,\mathrm{d}\boldsymbol{\lambda}_{l:l+1}. \quad (136)$$

Thus, we finally obtain, by subtracting both right hand sides of (135) and (136), we find, for $l=k-1$ or $l=k$

$$[\boldsymbol{P}_{12}]_{k,l} = u(\tau,h_l)\,\frac{\beta^{\alpha_l+\alpha_{l+1}}}{\Gamma(\alpha_l)\,\Gamma(\alpha_{l+1})}\int_{\mathbb{R}_+^2}\left(\max(\pm h_l,0)-\max(\mp h_l,0)\left(\left(\frac{\lambda_l}{\lambda_{l+1}}\right)^{\pm 1/2}-1\right)\right)\varphi_{h_l}(\boldsymbol{\lambda}_{l:l+1})\,\mathrm{d}\boldsymbol{\lambda}_{l:l+1}$$
$$= \pm h_l\,u(\tau,h_l)\,\frac{\beta^{\alpha_l+\alpha_{l+1}}}{\Gamma(\alpha_l)\,\Gamma(\alpha_{l+1})}\int_{\mathbb{R}_+^2}\left(\left(\frac{\lambda_l}{\lambda_{l+1}}\right)^{\pm 1/2}-1\right)\varphi_{h_l}(\boldsymbol{\lambda}_{l:l+1})\,\mathrm{d}\boldsymbol{\lambda}_{l:l+1} \quad (137)$$

where both "$\pm$" signs are "$+$" signs if $l=k-1$, and they are "$-$" signs if $l=k$. More explicitly, for $k=1,\ldots,K$,

$$[\boldsymbol{P}_{12}]_{k,k} = -h_k\,u(\tau,h_k)\,\frac{\beta^{\alpha_k+\alpha_{k+1}}}{\Gamma(\alpha_k)\,\Gamma(\alpha_{k+1})}\int_{\mathbb{R}_+^2}\left(\sqrt{\frac{\lambda_{k+1}}{\lambda_k}}-1\right)\varphi_{h_k}(\boldsymbol{\lambda}_{k:k+1})\,\mathrm{d}\boldsymbol{\lambda}_{k:k+1} \quad (138)$$

and

$$[\boldsymbol{P}_{12}]_{k+1,k} = h_k\,u(\tau,h_k)\,\frac{\beta^{\alpha_k+\alpha_{k+1}}}{\Gamma(\alpha_k)\,\Gamma(\alpha_{k+1})}\int_{\mathbb{R}_+^2}\left(\sqrt{\frac{\lambda_k}{\lambda_{k+1}}}-1\right)\varphi_{h_k}(\boldsymbol{\lambda}_{k:k+1})\,\mathrm{d}\boldsymbol{\lambda}_{k:k+1} \quad (139)$$

Finally, considering (134) and (137), we obtain the structure of the matrix $\boldsymbol{P}_{12}$ as given by (26) and its elements as given by (27).

We have thus completed the derivation of the bound given in Section IV.

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