Cramér-Rao Bound for Lie Group Parameter Estimation with Euclidean Observations and Unknown Covariance Matrix

Samy Labsir*, Sara El Bouch†, Alexandre Renaux§, Jordi Vilà-Valls‡ and Eric Chaumette‡
* IPSA/TéSA, Toulouse, France , † J-L. Lagrange, University of Nice, CNRS, Nice, France, ‡ ISAE-SUPAERO, University of Toulouse, Toulouse, France, § University of Paris-Saclay, Paris, France

***Abstract*—This article addresses the problem of computing a Cramér-Rao bound when the likelihood of Euclidean observations is parameterized by both unknown Lie group (LG) parameters and covariance matrix. To achieve this goal, we leverage the LG structure of the space of positive definite matrices. In this way, we can assemble a global LG parameter that lies on the product of the two groups, on which LG's intrinsic tools can be applied. From this, we derive an inequality on the intrinsic error, which can be seen as the equivalent of the Slepian-Bangs formula on LGs. Subsequently, we obtain a closed-form expression of this formula for Euclidean observations. The proposed bound is computed and implemented on two real-world problems involving observations lying in $\mathbb{R}^p$, dependent on an unknown LG parameter and an unknown noise covariance matrix: the Wahba's estimation problem on $SE(3)$, and the inference of the pose in $SE(3)$ of a camera from pixel detections.**

***Index Terms*—Cramér-Rao bound (CRB), Lie groups (LGs), unknown covariance matrix.**

I. Introduction

The accuracy of wide-sense unbiased estimators can be characterized by the Cramér-Rao bound (CRB), which gives insights on the ultimate minimum mean square error (MSE) [1]–[5]. The standard CRB formulation is suitable for unknown deterministic parameters lying in the Euclidean space. To tackle estimation problems involving unknown parameters that must conform to certain geometric constraints, it is crucial to develop new estimation metrics and associated CRBs, hereinafter referred to as *intrinsic*, that take into account their geometry. In particular, recent emphasis has been placed on cases where the parameters belong to a Lie group (LG), a parameter space commonly encountered in the fields of signal processing, robotics, and control theory [6], [7].

A myriad of practical estimation problems exemplify the use of parameters on LGs. To cite a few, in computer vision, the registration problems aim to find correspondences between two images, this mapping naturally translates to a geometric transformation in the *special Euclidean group*, denoted as LG $SE(3)$, or the *similarity group* $SL(3)$ [8]. Moreover, in numerous target tracking applications, it is often necessary to estimate the orientation of a mobile object constrained to lie on the LG of *unitary rotation matrices*, denoted as $SO(3)$ [9]. In line with the widespread practical estimation problems involving LGs, in the last decade, there has been a growing interest in deriving lower bounds on LGs, [10]–[12]. These lower bounds can be obtained by minimizing an intrinsic MSE (IMSE) consistent with the LG geometric structure.

In particular, in the seminal papers [12], [13], an inequality on the IMSE for LGs was proposed. Under certain assumptions, it is possible to approximate this inequality and obtain an analytical CRB expression on LGs (LG-CRB) for Euclidean observations, that only admits a closed-form expression for the specific LG $SO(3)$ [14]–[17]. To bypass these shortcomings, a new LG-CRB formulation was proposed in [18], which adapts the formalism of the Euclidean Barankin and McAulay-Seidman bounds [19]–[21] to LGs. This new formulation allows to derive the LG-CRB without approximations. Notably, this new bound admits a closed-form expression for the LG $SE(3)$ in the case of Euclidean observations.

Furthermore, many estimation problems on LGs assume, for simplicity, that the noise covariance on observations is known. However, this simplifying assumption does not hold in many real-world applications. For example, in the context of extended target tracking, radar observations are spread across different areas of the target and are characterized by an unknown dispersion covariance matrix that needs to be estimated [22], [23]. Likewise, in computer vision, a camera measures and detects pixels in an image, which depend on the unknown camera transformation in $SE(3)$, where the detection noise is usually unknown due to the image signal-to-noise ratio [24], [25]. These problems require the joint estimation of unknown deterministic parameters on LG and a covariance matrix. To evaluate the theoretical performance of this joint estimator, an intrinsic CRB must be developed.

In the state-of-the-art, the space of semi-positive definite matrices, such as covariance matrices, forms a differentiable manifold so-called Riemannian manifold. The Riemannian structure of the covariance matrix space is classically used for intrinsic estimation problems, and associated intrinsic CRBs exist [16], [26]. The derivations of the intrinsic CRBs are performed using metrics on the tangent space, depending on the considered observation model, particularly for Gaussian and elliptical distributions [27], [28]. Consequently, it is challenging to obtain a bound considering the Riemannian structure of both the covariance matrix space and the LG of interest, as the Riemannian metric of the latter is difficult to characterize, especially for most physical LGs such as $SE(3)$ or $SL(3)$, making derivations and integrals impossible to compute.

It is worth noting that covariance matrix estimation with LG parameters has already been addressed in [29], [30] by performing eigenvalue decomposition, which reveals the LG product of a diagonal matrix and $SO(s)$. While this approach may be relevant for estimation algorithms, it does not allow for the computation of LG-CRB analytical expressions. To overcome this limitation, it is possible to leverage the so-called log-Euclidean framework [31], [32] for the set of symmetric positive definite matrices (SPD), denoted as $\mathcal{P}^+(s)$. Equipped with the log-Euclidean law, $\mathcal{P}^+(s)$ defines a commutative LG [33]. Its LG structure offers several advantages. First, it provides a unified formalism for the estimation problem on an augmented LG when dealing with unknown parameters belonging to a LG. Second, it allows for computations that lead to tractable expressions for various observation models. Using the LG properties can be relevant for two reasons: first, any LG equipped with a law can also be equipped with a metric specific to its LG structure, and second, its metric enables handling derivative computations [31], [34].

The main contribution of this article is to formalize a LG-CRB on LGs, called Covariance LG CRB (C-LG-CRB), by incorporating the covariance matrix into the unknown parameters with its LG structure for Euclidean observations. To formalize the C-LG-CRB while taking into account the covariance matrix, we exploit the fact that if the unknown parameters of interest belong to a LG G, then the tuple of unknown parameters, gathering the latter and the unknown covariance matrix, belongs to the LG product $G \times \mathcal{P}^+(s)$. By using the intrinsic properties and tools of this new LG space, we can derive a generic bound expression. Another contribution is to derive this bound for two important examples with Euclidean observation models. First, we consider the case where 3D observations are available, which depend on an unknown affine transformation, as in the Wahba's problem [35]. Second, we consider an application based on the well-known pin-hole model for a sensor camera observing a scene and detecting obstacles [36]. In this context, 2D observations depend on two affine transformations linked to the camera pose. To the best of our knowledge, the LG-CRB has been derived for the first model but not for the second one. Moreover, the C-LG-CRB has not been computed for either of the two models. Numerical results are provided to support the discussion.

II. Background on LGs Estimation

A. Matrix Lie group: properties

A matrix LG $(G \subset \mathbb{R}^{n\times n}, \circledast)$ is a matrix space equipped with a structure of both smooth manifold and group. Its structure of group defines an internal law $\circledast$ and its structure of smooth manifold defines a tangent space at each point of G. The identity tangent space $\mathfrak{g}$ is the Lie algebra where each element is connected locally to each element of G through the logarithm and exponential applications defined, respectively, by $\text{Exp}_G : \mathfrak{g} \to G$ and $\text{Log}_G : G \mapsto \mathfrak{g}$, as illustrated in figure 1. As $\mathfrak{g}$ is isomorph to $\mathbb{R}^m$, we can define two bijections $[.]^\wedge : \mathbb{R}^m \mapsto \mathfrak{g}$ and $[.]^\vee : \mathfrak{g} \mapsto \mathbb{R}^m$. m is the dimension of the Lie algebra. In this way, we can denote the exponential and logarithm applications such as $\forall\, \mathbf{a} \in \mathbb{R}^m$, $\text{Exp}_G^\wedge(\mathbf{a}) = \text{Exp}\left([\mathbf{a}]_G^\wedge\right)$ and $\forall\, \mathbf{M} \in G, [\text{Log}_G(\mathbf{M})]_G^\vee = \text{Log}_G^\vee(\mathbf{M})$. For more details about the LG theory, the reader can refer to [37], [38]. As a LG is connected to tangent spaces, we can define the notion of derivation. Especially, it is feasible to generalize the definition of the Euclidean directional derivative on LGs by defining a direction in the Lie algebra. Let $\mathbf{g} : G \mapsto H$ be a LG-valued function in H with law $\oslash$. The right Lie derivative of $\mathbf{g}$ in $\mathbf{M} \in G$ is given by ($\forall \boldsymbol{\epsilon} \in \mathbb{R}^m$):

$$\mathcal{L}^R_{\mathbf{g}(\mathbf{M})} = \left.\frac{\partial \text{Log}_H^\vee\left(\mathbf{g}(\mathbf{M})^{-1} \oslash \mathbf{g}(\mathbf{M} \circledast \text{Exp}_G^\wedge(\boldsymbol{\epsilon})\,)\right)}{\partial \boldsymbol{\epsilon}^\top}\right|_{\boldsymbol{\epsilon}=\mathbf{0}}. \tag{1}$$

When $\mathbf{g}$ has values in $\mathbb{R}$ or $\mathbb{R}^p$ then $\text{Log}_H^\vee(.) = \mathbf{I}$ and the derivative of $\mathbf{g}$ can be defined as [39]:

$$\mathcal{L}^R_{\mathbf{g}(\mathbf{M})} = \left.\frac{\partial \mathbf{g}(\mathbf{M} \circledast \text{Exp}_G^\wedge(\boldsymbol{\epsilon})\,)}{\partial \boldsymbol{\epsilon}^\top}\right|_{\boldsymbol{\epsilon}=\mathbf{0}} \quad \forall \boldsymbol{\epsilon} \in \mathbb{R}^m. \tag{2}$$

$$\triangleq \boldsymbol{\nabla}\mathbf{g}(\mathbf{M}, \circledast) \tag{3}$$

It is worth noting that this definition of the Lie group gradient can be related to the Riemannian gradient [40, proposition 3.59], due to the Riemannian structure of a Lie group.

In addition, if $\mathbf{M} = \begin{bmatrix}\mathbf{M}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{M}_2\end{bmatrix}$ with $\mathbf{M}_1 \in G_1$ with law $*$ then the Lie derivative of $\mathbf{g}$ according to $\mathbf{M}_1$ is:

$$\mathcal{L}^R_{\mathbf{g}(\mathbf{M})} = \left.\frac{\partial \mathbf{g}(\mathbf{M} \circledast \text{Exp}_G^\wedge\left([\boldsymbol{\epsilon}_{\mathbf{M}_1}^\top, \mathbf{0}^\top]\right))}{\partial \boldsymbol{\epsilon}_{\mathbf{M}_1}^\top}\right|_{\boldsymbol{\epsilon}=\mathbf{0}} \tag{4}$$

$$\triangleq \boldsymbol{\nabla}_{\mathbf{M}_1}\mathbf{g}(\mathbf{M}, *) \tag{5}$$

It is also possible to define a left Lie derivative by switching $\mathbf{M}$ and $\text{Exp}_G^\wedge(\boldsymbol{\epsilon})$. In this work, we have chosen to use the right formalism, as it is more compatible with the LG-CRB formalism in the state-of-the-art [13] [14] [18], especially because it facilitates the computation of error bound on LGs in a Bayesian context [41].

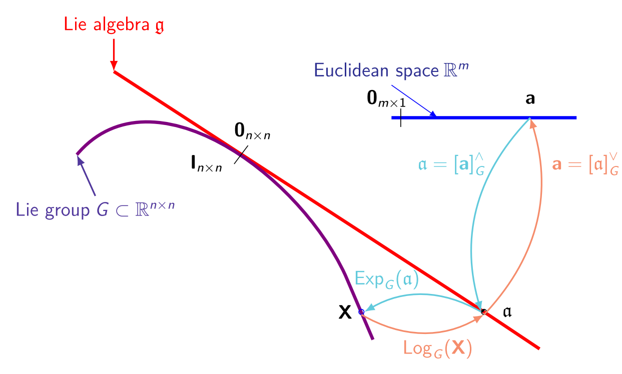


Fig. 1. Relation between $\mathbb{R}^m$, G and $\mathfrak{g}$.

B. Estimation and Cramér-Rao bound on Lie groups

To define estimation concepts, and in particular the concept of estimation error, we first need to define the tool for intrinsically quantifying the gap between two LG points through the intrinsic path.

1) Notion of intrinsic path:

Definition II-B.1 (Intrinsic path on LGs)**.** *The intrinsic path between two LG elements* $\mathbf{A}$ *and* $\mathbf{B}$ *is defined as*

$$\boldsymbol{l}_G^{\circledast}(\mathbf{A},\mathbf{B}) = \mathrm{Log}_G^{\vee}\left(\mathbf{A}^{-1} \circledast \mathbf{B}\right) \ , \ \forall\ (\mathbf{A},\mathbf{B}) \in G \times G. \quad (6)$$

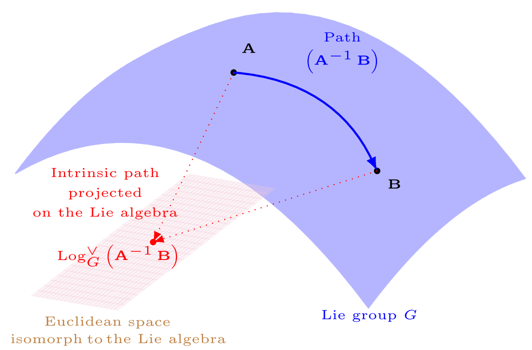


Fig. 2. Illustration of the intrinsic path between two points $\mathbf{A}$ and $\mathbf{B} \in G$. The path between $\mathbf{A}$ and $\mathbf{B}$ is given by $\mathbf{A}^{-1} * \mathbf{B}$ and is projected on the Lie algebra.

Remark:

It is worth pointing out that the norm of this intrinsic path, the gap $\|\boldsymbol{l}_G(\mathbf{A},\mathbf{B})\|^2$ does not generally define a distance mathematically speaking (although this is the case for the LG $SO(3)$), because not all axioms of a distance are respected. Nevertheless, as illustrated in figure 2, it specifies a good indicator of the intrinsic path traveled between two LG points, and it is classically used in the LG estimation literature [6] [42] [43] to compare an unknown LG parameter with its estimator.

2) Estimation on Lie groups: We consider the case where the parameter $\mathbf{M}$ lies on the LG G equipped with the law $\circledast$. We consider a set of Euclidean observations $\mathbf{z} = \{\mathbf{z}_i\}_{i=1}^N$ with $\mathbf{z}_i \in \mathbb{R}^s$ distributed according to the likelihood $p(\mathbf{z}|\mathbf{M})$. An estimator of $\mathbf{M}$, denoted $\widehat{\mathbf{M}}$, is described by the same indicators as in the Euclidean space, but in an intrinsic way [9] [44]. Indeed, according to the definition (II-B.1), the intrinsic bias $\mathbf{b}_{\mathbf{z}|\mathbf{M}}(\mathbf{M},\widehat{\mathbf{M}})$, the mean $\mathbf{M}_{\widehat{\mathbf{M}}}$ and the IMSE $\mathbf{C}_{\mathbf{z}|\mathbf{M}}(\mathbf{M},\widehat{\mathbf{M}},\circledast)$ can be respectively defined by:

$$\mathbf{b}_{\mathbf{z}|\mathbf{M}}(\mathbf{M},\widehat{\mathbf{M}}) = \mathbb{E}_{p(\mathbf{z}|\mathbf{M})}\left(\boldsymbol{l}_G^{\circledast}\left(\mathbf{M},\widehat{\mathbf{M}}\right)\right), \quad (7)$$

$$\mathbb{E}_{p(\mathbf{z}|\mathbf{M})}\left(\boldsymbol{l}_G^{\circledast}\left(\widehat{\mathbf{M}},\mathbf{M}_{\widehat{\mathbf{M}}}\right)\right) = \mathbf{0}, \quad (8)$$

$$\mathbf{C}_{\mathbf{z}|\mathbf{M}}(\mathbf{M},\widehat{\mathbf{M}},\circledast) = \mathbb{E}_{p(\mathbf{z}|\mathbf{M})}\left(\boldsymbol{l}_G^{\circledast}\left(\mathbf{M},\widehat{\mathbf{M}}\right)\boldsymbol{l}_G^{\circledast}\left(\mathbf{M},\widehat{\mathbf{M}}\right)^{\top}\right). \quad (9)$$

Knowing these different indicators, it is possible to formalize an intrinsic error bound for $\mathbf{M} \in G$ which is a lower bound on the IMSE, as illustrated in figure 3. In the following, for the sake of clarity, we define $\mathbb{E}(\cdot) \triangleq \mathbb{E}_{p(\mathbf{z}|\mathbf{M})}(\cdot)$.

3) Cramér-Rao bound on Lie groups: The first version of the CRB on LGs was proposed in the seminal paper [11]. However, the proposed bound only admits a closed-form for the group of rotations $SO(n)$. To bypass this shortcoming, we have proposed in our previous work [18] a more general CRB formulation on LGs. For the purpose of this article, we will make use of this alternative formulation.

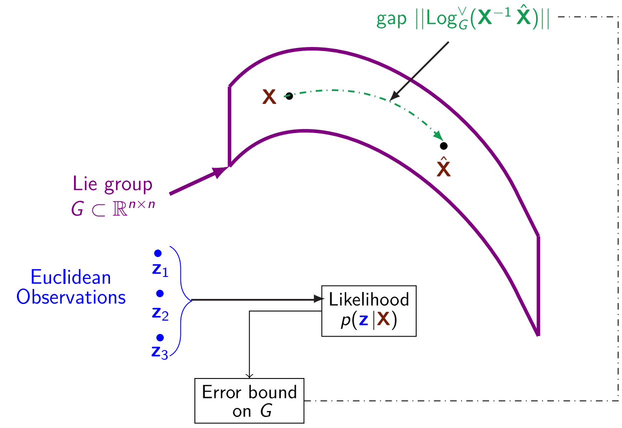


Fig. 3. Building an error bound on Lie groups with Euclidean observations.

Theorem II-B.1 (LG-CRB for Gaussian Euclidean observations). *By assuming that we can build an unbiased estimator* $\widehat{\mathbf{M}}$*, in the sense of (7), from observations* $\mathbf{z} = \{\mathbf{z}_i\}_{i=1}^N$*,* $\mathbf{z}_i \in \mathbb{R}^s$*, with likelihood* $p(\mathbf{z}|\mathbf{M})$ *verifying the following assumption:*

$$\int \boldsymbol{\nabla}\left(\log p(\mathbf{z}|\mathbf{M}),\circledast\right)\mathrm{d}\mathbf{z} = \nabla\left(\int \log p(\mathbf{z}|\mathbf{M})\mathrm{d}\mathbf{z},\circledast\right) \quad (10)$$

then, the IMSE is bounded by the LG-CRB[1]*:*

$$\mathbf{C}_{\mathbf{z}|\mathbf{M}}(\mathbf{M},\widehat{\mathbf{M}},\circledast) \succeq \mathcal{I}_G^{-1}, \quad (11)$$

where $\mathcal{I}_G$ *is the intrinsic Fisher information matrix (IFIM):*

$$\mathcal{I}_G = \mathbb{E}\left(\nabla lp(\mathbf{M},\circledast)\nabla lp(\mathbf{M},\circledast)^{\top}\right), \quad (12)$$

and $lp(\mathbf{M},\boldsymbol{\delta},\circledast) = \log p(\mathbf{z}|\mathbf{M} \circledast \mathrm{Exp}_G^{\wedge}(\boldsymbol{\delta}))$ *which can be rewritten:*

$$\mathcal{I}_G = -\mathbb{E}(\,\boldsymbol{\nabla}^2 lp(\mathbf{M},\circledast)\,), \quad (13)$$

where

$$\boldsymbol{\nabla}^2 lp(\mathbf{M},\circledast) = \left.\frac{\partial^2 \log p(\mathbf{z}|\mathbf{M} \circledast \mathrm{Exp}_G^{\wedge}(\boldsymbol{\delta}_1) \circledast \mathrm{Exp}_G^{\wedge}(\boldsymbol{\delta}_2))}{\partial\boldsymbol{\delta}_1\partial\boldsymbol{\delta}_2^{\top}}\right|_{\boldsymbol{\delta}_1=\boldsymbol{\delta}_2=\mathbf{0}} \quad (14)$$

Remark:

It is important to stress that the LG-CRB is different from the Riemannian CRB proposed in [46] because it is intrinsic to its group structure. Indeed, this formalism does not need to use a Riemannian metric, specific to the Riemannian structure of the considered LG, but only the notion of an intrinsic path defined by equation (6).

Corollary II-B.1.1 (Expression of LG-CRB for Gaussian Euclidean observations). *Let us assume that* $p(\mathbf{z}_i|\mathbf{M})$ *is a Gaussian distribution:*

$$p(\mathbf{z}_i|\mathbf{M},\boldsymbol{\Sigma}) = \mathcal{N}(\mathbf{z}_i,\mathbf{f}_i(\mathbf{M}),\boldsymbol{\Sigma}), \quad (15)$$

[1]If we consider two matrices $\boldsymbol{\Sigma}_1$, $\boldsymbol{\Sigma}_2$, $\boldsymbol{\Sigma}_2 \succeq \boldsymbol{\Sigma}_1$ means that the matrix $\boldsymbol{\Sigma}_2$ - $\boldsymbol{\Sigma}_1$ is positive (Löwner ordering [45]).

where $\mathbf{f}_i : G \to \mathbb{R}^s$ is a smooth function and $\mathbf{\Sigma} \in \mathbb{R}^{s\times s}$. If $\{\mathbf{z}_i\}_{i=1}^N$ are independent between them, then the matrix $\mathcal{I}_G$ verifies the following relation on LGs,

$$\mathcal{I}_G = \sum_{i=1}^{N} \boldsymbol{\nabla}\mathbf{f}_i(\mathbf{M}, \circledast)^\top \mathbf{\Sigma}^{-1}\, \nabla\mathbf{f}_i(\mathbf{M}, \circledast). \quad (16)$$

To draw a parallel with well-known established results in parametric estimation problems for Euclidean parameters, the Slepian-Bangs formula [47], [48] provides a closed-form expression (devoid of the expectation operator) of the Fisher information matrix for Gaussian observations with unknown mean and covariance. It is worth noting that the formula (16) can be seen as an intrinsic equivalent of the Slepian-Bangs formula in the case where the covariance matrix is known. In this work, we propose to extend this formula by integrating it into the unknown parameter space.

III. Development of the Cramér-Rao Bound with Unknown Covariance Matrix

In this section, we first give the tools to define the set of covariance matrices as a LG. More precisely, we introduce its intrinsic properties. Then, we express and detail the expression of the proposed C-LG-CRB taking into account an unknown covariance matrix. Particularly, we provide a generic expression, and a specific expression for the Gaussian Euclidean model depending on both unknown LG parameters and covariance matrix.

A. Lie group product and SPD matrices

Theorem III-A.1 (Lie group structure of SPD matrices). *The set of covariance matrices with size s forms the space of symmetric positive definite (SPD) matrices $\mathcal{P}^+(s)$ defined such as $\forall \mathbf{\Sigma} \in \mathcal{P}^+(s)$*

$$|\mathbf{\Sigma}| > 0, \quad \mathbf{\Sigma} = \mathbf{\Sigma}^\top. \quad (17)$$

Let us define Expm *and* Logm *respectively as the exponential and matrix logarithm*. The group $(\mathcal{P}^+(s), \odot, \mathbf{I})$ defined by:

$$\mathbf{\Sigma}_1 \odot \mathbf{\Sigma}_2 = \text{Expm}\left(\text{Logm}(\mathbf{\Sigma}_1) + \text{Logm}(\mathbf{\Sigma}_2)\right) \ \forall\, (\mathbf{\Sigma}_1, \mathbf{\Sigma}_2) \in \mathcal{P}^+(s)^2, \quad (18)$$

is a commutative LG with Lie algebra $S^+(s)$ being the set of symmetric matrices with dimension $d = \frac{s(s+1)}{2}$ [31] [33]. Group logarithm and exponential are given by

$$\text{Log}^\vee_{\mathcal{P}^+(s)}(\mathbf{\Sigma}) = \textit{vech}\left(\text{Logm}(\mathbf{\Sigma})\right), \quad (19)$$

$$\text{Exp}^\wedge_{\mathcal{P}^+(s)}(\boldsymbol{\delta}) \ = \text{Expm}(\textit{sym}\,(\boldsymbol{\delta})), \quad (20)$$

where the operator vech stacks the columns of the matrix Logm$(\mathbf{\Sigma}) \in S^+(s)$ one below another into a single column vector by excluding the upper triangular portion. sym is the reciprocal function of vech. Furthermore, the intrinsic path between two elements is:

$$\boldsymbol{l}^{\odot}_{\mathcal{P}^+(s)}(\mathbf{\Sigma}_1, \mathbf{\Sigma}_2) \triangleq Log^\vee_{\mathcal{P}^+(s)}(\mathbf{\Sigma}_1) \ - Log^\vee_{\mathcal{P}^+(s)}(\mathbf{\Sigma}_2)\ .$$

The latter can be interpreted as the Euclidean path computing the difference between the two points projected on the Lie algebra. In the case where $s = 2$ and $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are diagonals, it amounts to compute the gap between the real logarithm of each diagonal coefficient.

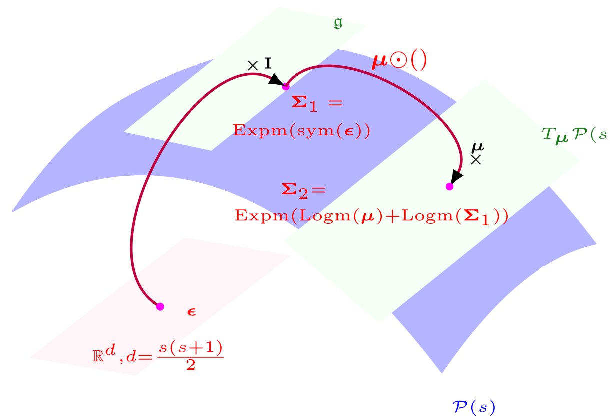


Fig. 4. Building an element of $\mathcal{P}^+(s)$. An element of $\mathbb{R}^d$ is brought to the tangent space at identity $S^+(s)$ through the operator *sym*. Then, the associated element $\mathbf{\Sigma}_1 \in \mathcal{P}^+(s)$ is obtained by applying Expm.

<u>*Remark:*</u>

If we consider a LG-defined function $\mathbf{h} : \mathcal{P}^+(s) \to \mathbb{R}^l$. Its Lie derivative according to $\mathbf{\Sigma} \in \mathcal{P}^+(s)$ is given by the formula (3) and is written as $\forall \mathbf{\Sigma} \in \mathcal{P}^+(s)$:

$$\mathcal{L}^R_{\mathbf{h}(\mathbf{\Sigma})} = \left.\frac{\partial \mathbf{h}(\mathbf{\Sigma} \odot \text{Exp}^\wedge_{\mathcal{P}^+(s)}(\boldsymbol{\delta})\ )}{\partial \boldsymbol{\delta}^\top}\right|_{\boldsymbol{\delta}=\mathbf{0}} \quad (21)$$

$$\triangleq \left.\frac{\partial \mathbf{h}(\text{Expm}\left(\text{Logm}(\mathbf{\Sigma}) + \textit{sym}\,(\boldsymbol{\delta})\right))}{\partial \boldsymbol{\delta}^\top}\right|_{\boldsymbol{\delta}=\mathbf{0}} \quad (22)$$

Theorem III-A.2 (Lie group product with SPD matrices). *Now, let us consider a matrix LG $(G \subset \mathbb{R}^{n\times n}, *)$ with dimension g. The Lie group product $G' = G \times \mathcal{P}^+(s)$ is also a LG equipped with law $\oplus$ such as $\forall\ \mathbf{X}_1 = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0}_{n\times s} \\ \mathbf{0}_{s\times n} & \mathbf{\Sigma}_1 \end{bmatrix}$ $\in G \times \mathcal{P}^+(s), \forall\ \mathbf{X}_2 = \begin{bmatrix} \mathbf{M}_2 & \mathbf{0}_{n\times s} \\ \mathbf{0}_{s\times n} & \mathbf{\Sigma}_2 \end{bmatrix} \in G \times \mathcal{P}^+(s)$:*

$$\mathbf{X}_1 \oplus \mathbf{X}_2 = \begin{bmatrix} \mathbf{M}_1 * \mathbf{M}_2 & \mathbf{0}_{n\times s} \\ \mathbf{0}_{s\times n} & \mathbf{\Sigma}_1 \odot \mathbf{\Sigma}_2. \end{bmatrix} \quad (23)$$

The group exponential application of G' is given $\forall \boldsymbol{\delta} = \left[\boldsymbol{\delta}_G^\top, \boldsymbol{\delta}_P^\top\right]^\top \in \mathbb{R}^{g+d}$ by

$$\text{Exp}^\wedge_{G'}(\boldsymbol{\delta}) = \begin{bmatrix} \text{Exp}^\wedge_G(\boldsymbol{\delta}_G) & \mathbf{0}_{n\times s} \\ \mathbf{0}_{s\times n} & \text{Exp}^\wedge_{\mathcal{P}^+(s)}(\boldsymbol{\delta}_P) \end{bmatrix}. \quad (24)$$

Concerning the logarithm application, it is given by

$$\text{Log}^\vee_{G'}(\mathbf{X}) = \begin{bmatrix} \text{Log}^\vee_G(\mathbf{M}) \\ \text{Log}^\vee_{\mathcal{P}^+(s)}(\mathbf{\Sigma}) \end{bmatrix} \forall\, \mathbf{X} = \begin{bmatrix} \mathbf{M} & \mathbf{0}_{n\times s} \\ \mathbf{0}_{s\times n} & \mathbf{\Sigma} \end{bmatrix} \quad (25)$$

and the intrinsic path by

$$\boldsymbol{l}^{\oplus}_{G'}(\mathbf{X}_1, \mathbf{X}_2) = \left[\boldsymbol{l}^{*}_{G}(\mathbf{M}_1, \mathbf{M}_2)^\top, \boldsymbol{l}^{\odot}_{\mathcal{P}^+(s)}(\mathbf{\Sigma}_1, \mathbf{\Sigma}_2)^\top\right]^\top.$$

B. Expression of the bound

In this subsection, we provide the expression of the proposed intrinsic bound (C-LG-CRB).

Corollary III-B.0.1 (C-LG-CRB for a generic model). *We consider a set of observations* $\mathbf{z} = \{\mathbf{z}_i\}_{i=1}^N, \mathbf{z}_i \in \mathbb{R}^s$ *characterized by its likelihood* $p(\mathbf{z}|\mathbf{X})$ *where the unknown parameter is* $\mathbf{X} \in G' = G \times \mathcal{P}^+(s)$*. According to* (III-A.1) *and* (III-A.2)*, the IMSE between* $\mathbf{X}$ *and* $\widehat{\mathbf{X}}$ *is bounded by* $\mathcal{I}_{G'}^{-1}$.

Theorem III-B.1 (C-LG-CRB for Gaussian Euclidean observations). *If the observations* $\mathbf{z} = \{\mathbf{z}_1, \dots, \mathbf{z}_N\}$ *are independently distributed from*

$$p(\mathbf{z}_i|\mathbf{M}, \mathbf{\Sigma}) = \mathcal{N}(\mathbf{z}_i; \mathbf{f}_i(\mathbf{M}), \mathbf{\Sigma}), \tag{26}$$

then

$$\mathcal{I}_{G'} = \begin{bmatrix} \mathbf{A} & \mathbf{0}_{d\times g} \\ \mathbf{0}_{g\times d} & \mathbf{B} \end{bmatrix}, \tag{27}$$

where

$$\mathbf{A} = \sum_{i=1}^N \boldsymbol{\nabla}\mathbf{f}_i(\mathbf{M}, *)^\top \mathbf{\Sigma}^{-1} \boldsymbol{\nabla}\mathbf{f}_i(\mathbf{M}, *) \tag{28}$$

$$\mathcal{L}^R_{\mathbf{f}_i(\mathbf{M})} = \left.\frac{\partial \mathbf{f}_i(\mathbf{X} * \mathrm{Exp}^\wedge_G(\boldsymbol{\epsilon}_1^{\mathbf{M}}))}{\partial \left(\boldsymbol{\epsilon}_1^{\mathbf{M}}\right)^\top}\right|_{\boldsymbol{\epsilon}_1^{\mathbf{M}}=\mathbf{0}} \tag{29}$$

$$\mathbf{B} = \frac{N}{2}\mathbf{T} \tag{30}$$

$$\mathbf{T} = \mathrm{diag}\left[\underbrace{1, \dots, 1}_{s}, \underbrace{2, \dots, 2}_{\frac{s\,(s-1)}{2}}\right]. \tag{31}$$

Proof:

As the unknown parameter are divided into two parts, we can decompose $\mathcal{I}_{G'}$ as follows:

$$\mathcal{I}_{G'} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{bmatrix} \tag{32}$$

with

$$\mathbf{A} = \mathbb{E}\left(\boldsymbol{\nabla}_{\mathbf{M}} lp(\mathbf{X}, \oplus)\, \boldsymbol{\nabla}_{\mathbf{M}} lp(\mathbf{X}, \oplus)^\top\right) \tag{33}$$

$$\mathbf{B} = \mathbb{E}\left(\boldsymbol{\nabla}_{\mathbf{\Sigma}} lp(\mathbf{X}, \oplus)\, \boldsymbol{\nabla}_{\mathbf{\Sigma}} lp(\mathbf{X}, \oplus)^\top\right) \tag{34}$$

$$\mathbf{C} = \mathbb{E}\left(\boldsymbol{\nabla}_{\mathbf{M}} lp(\mathbf{X}, \oplus)\, \boldsymbol{\nabla}_{\mathbf{\Sigma}} lp(\mathbf{X}, \oplus)^\top\right). \tag{35}$$

• Computation of $\mathbf{A}$:

By applying the Slepian-Bangs formula (16) on $\mathbf{M}$, we obtain directly:

$$\mathbf{A} = \sum_{i=1}^N \boldsymbol{\nabla}\mathbf{f}_i(\mathbf{M}, *)^\top \mathbf{\Sigma}^{-1} \boldsymbol{\nabla}\mathbf{f}_i(\mathbf{M}, *). \tag{36}$$

• Computation of $\mathbf{B}$:

As $\boldsymbol{\epsilon} \to \log p\left(\mathbf{Z}|\mathbf{X} \oplus \mathrm{Exp}^\wedge_G(\boldsymbol{\epsilon})\right)$ is a quadratic function, it verifies (10)

$$\mathbb{E}\left(\boldsymbol{\nabla}_{\mathbf{\Sigma}} lp(\mathbf{X}, \oplus)\, \boldsymbol{\nabla}_{\mathbf{\Sigma}} lp(\mathbf{X}, \oplus)^\top\right) = -\mathbb{E}\left(\boldsymbol{\nabla}^2_{\mathbf{\Sigma}} lp(\mathbf{X}, \oplus)\right). \tag{37}$$

On the other hand, we know that:

$$\begin{aligned}\boldsymbol{\nabla}^2_{\mathbf{\Sigma}} lp(\mathbf{X}, \oplus) = {}& \frac{\partial^2}{\partial \boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}\, \partial \boldsymbol{\epsilon}_2^{\mathbf{\Sigma}}}\Bigg(-\frac{N}{2}\log|\mathbf{\Sigma} \odot \mathrm{Exp}^\wedge_{\mathcal{P}+(s)}\left(\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}\right) \odot \mathrm{Exp}^\wedge_{\mathcal{P}+(s)}\left(\boldsymbol{\epsilon}_2^{\mathbf{\Sigma}}\right)| - \\ & \frac{1}{2}\sum_{i=1}^N (\mathbf{z}_i - \mathbf{f}_i(\mathbf{M}))^\top (\mathbf{\Sigma} \odot \mathrm{Exp}^\wedge_{\mathcal{P}+(s)}\left(\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}\right) \odot \mathrm{Exp}^\wedge_{\mathcal{P}+(s)}\left(\boldsymbol{\epsilon}_2^{\mathbf{\Sigma}}\right))^{-1} \\ & (\mathbf{z}_i - \mathbf{f}_i(\mathbf{M}))|_{\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}=\mathbf{0}, \boldsymbol{\epsilon}_2^{\mathbf{\Sigma}}=\mathbf{0}}.\end{aligned} \tag{38}$$

and can be simplified in the following way:

- first, we demonstrate, in appendix B that:

$$\frac{\partial^2}{\partial \boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}\, \partial \boldsymbol{\epsilon}_2^{\mathbf{\Sigma}}}\log|\mathbf{\Sigma} \odot \mathrm{Exp}^\wedge_{\mathcal{P}+(s)}\left(\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}\right) \odot \mathrm{Exp}^\wedge_{\mathcal{P}+(s)}\left(\boldsymbol{\epsilon}_2^{\mathbf{\Sigma}}\right)| = \mathbf{0}, \tag{39}$$

- second, we show that $\forall(k,l) \in \{1, \dots, d\}^2$:

$$\begin{aligned}&\left[\frac{\partial^2}{\partial \boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}\, \partial \boldsymbol{\epsilon}_2^{\mathbf{\Sigma}}}(\mathbf{\Sigma} \odot \mathrm{Exp}^\wedge_{\mathcal{P}+(s)}\left(\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}\right) \odot \mathrm{Exp}^\wedge_{\mathcal{P}+(s)}\left(\boldsymbol{\epsilon}_2^{\mathbf{\Sigma}}\right))^{-1}\right]_{k,l} \\ &= -\mathbf{G}_k\, \mathbf{G}_l\, \mathrm{Expm}\left(-\mathrm{Logm}(\mathbf{\Sigma}) + sym(\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}) + sym(\boldsymbol{\epsilon}_2^{\mathbf{\Sigma}})\right),\end{aligned} \tag{40}$$

  where $\{\mathbf{G}_i\}_{i=1}^d$ is the natural basis of $S^+(n)$. Consequently, we obtain that $\forall(k,l) \in \{1, \dots, d\}^2$:

$$\left.\left[\boldsymbol{\nabla}^2_{\mathbf{\Sigma}} lp(\mathbf{X}, \oplus)\right]_{k,l}\right|_{\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}=\mathbf{0}, \boldsymbol{\epsilon}_2^{\mathbf{\Sigma}}=\mathbf{0}}$$

$$= -\frac{1}{2}\sum_{i=1}^N \mathrm{tr}\left((\mathbf{z}_i - \mathbf{f}_i(\mathbf{M}))^\top \mathbf{G}_k\, \mathbf{G}_l\, \mathbf{\Sigma}^{-1}\, (\mathbf{z}_i - \mathbf{f}_i(\mathbf{M}))\right) \tag{41}$$

$$= -\frac{1}{2}\sum_{i=1}^N \mathrm{tr}\left((\mathbf{z}_i - \mathbf{f}_i(\mathbf{M})(\mathbf{z}_i - \mathbf{f}_i(\mathbf{M}))^\top \mathbf{G}_k\, \mathbf{G}_l\, \mathbf{\Sigma}^{-1})\right), \tag{42}$$

and

$$\begin{aligned}&\mathbb{E}\left.\left[\boldsymbol{\nabla}^2_{\mathbf{\Sigma}} lp(\mathbf{X}, \oplus)\right]_{k,l}\right|_{\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}=\boldsymbol{\epsilon}_2^{\mathbf{\Sigma}}=\mathbf{0}} = \\ &-\frac{1}{2}\sum_{i=1}^N \mathrm{tr}\left(\mathbb{E}\left((\mathbf{z}_i - \mathbf{f}_i(\mathbf{M}))(\mathbf{z}_i - \mathbf{f}_i(\mathbf{M}))^\top\right) \mathbf{G}_k\, \mathbf{G}_l\, \mathbf{\Sigma}^{-1}\right) \\ &= -\frac{1}{2}\sum_{i=1}^N \mathrm{tr}\left(\mathbf{\Sigma}\, \mathbf{G}_k\, \mathbf{G}_l\, \mathbf{\Sigma}^{-1}\right) \\ &= -\frac{1}{2}\, N\, \mathrm{tr}\left(\mathbf{G}_k\, \mathbf{G}_l\right).\end{aligned}$$

Finally, we obtain that $\mathbf{B} = \frac{N}{2}\mathbf{T}$, where

$$(\mathbf{T})_{k,l} = \mathrm{tr}\left(\mathbf{G}_k\, \mathbf{G}_l\right) \quad \forall(k,l) \in \{1, \dots, d\}^2. \tag{43}$$

Here, it is worth noting that the highlighted expression of $\mathbf{B}$ depends on the order in which the base is selected through the matrix $\mathbf{T}$. Nonetheless, this order is just a convention, not changing the IFIM calculation but only the order of the d parameters to be stacked. If we choose the convention proposed in the appendix (A-B), one obtains:

$$\mathbf{T} = \text{diag}\left[\underbrace{1,\ldots,1}_{n},\underbrace{2,\ldots,2}_{\frac{n\,(n-1)}{2}}\right]. \quad (44)$$

• Computation of $\mathbf{C}$:

In the same way as previously, if $\boldsymbol{\epsilon}\to\log p\left(\mathbf{Z}|\mathbf{X}\oplus\text{Exp}^{\wedge}_{G'}(\boldsymbol{\epsilon})\right)$ is sufficiently regular in the sense of the condition (10) then

$$\begin{aligned}&\mathbb{E}\left(\boldsymbol{\nabla}_{\mathbf{M}}lp(\mathbf{X},\oplus)\,\boldsymbol{\nabla}_{\boldsymbol{\Sigma}}lp(\mathbf{X},\oplus)^{\top}\right)\\ &= -\,\mathbb{E}\left(\left.\frac{\partial^2 lp\left(\mathbf{X},\left[\boldsymbol{\epsilon}_1^{\mathbf{M}};\mathbf{0}\right],\left[\mathbf{0};\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}\right],\oplus\right)}{\partial\boldsymbol{\epsilon}_1^{\mathbf{M}}\,\partial\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}}\right|_{\boldsymbol{\epsilon}_1^{\mathbf{M}},\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}=\mathbf{0}}\right) \quad (45)\\ &= -\mathbb{E}\left(\boldsymbol{\nabla}_{\mathbf{M},\boldsymbol{\Sigma}}lp(\mathbf{X},\oplus)\right). \quad (46)\end{aligned}$$

Furthermore, we can demonstrate that:

$$\begin{aligned}\boldsymbol{\nabla}_{\mathbf{M}}lp(\mathbf{X},\oplus) = &\frac{\partial\mathbf{f}_i(\mathbf{M}*\text{Exp}^{\wedge}_{G}(\boldsymbol{\epsilon}_1^{\mathbf{M}}))}{\partial\boldsymbol{\epsilon}_1^{\mathbf{M}}}^{\top}\boldsymbol{\Sigma}^{-1}\\ &\left.\left(\mathbf{z}_i-\mathbf{f}_i(\mathbf{M}*\text{Exp}^{\wedge}_{G}(\boldsymbol{\epsilon}_1^{\mathbf{M}})\right)\right|_{\boldsymbol{\epsilon}_1^{\mathbf{M}}=\mathbf{0}}. \quad (47)\end{aligned}$$

Therefore, by differentiating the previous expression according to $\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}$, we gather:

$$\begin{aligned}&\boldsymbol{\nabla}^2_{\mathbf{M},\boldsymbol{\Sigma}}lp(\mathbf{X},\oplus)\\ &= \frac{\partial\mathbf{f}_i(\mathbf{M}*\text{Exp}^{\wedge}_{G}(\boldsymbol{\epsilon}_1^{\mathbf{M}}))}{\partial\boldsymbol{\epsilon}_1^{\mathbf{M}}}^{\top}\frac{\partial\left(\boldsymbol{\Sigma}\odot\text{Exp}^{\wedge}_{\mathcal{P}^+(s)}(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}})\right)^{-1}}{\partial\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}}\\ &\quad\times\left.\left(\mathbf{z}_i-\mathbf{f}_i(\mathbf{M}*\text{Exp}^{\wedge}_{G}(\boldsymbol{\epsilon}_1^{\mathbf{M}})\right)\right)\right|_{\boldsymbol{\epsilon}_1^{\mathbf{M}}=\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}=\mathbf{0}}.\\ &= \left.\frac{\partial\mathbf{f}_i(\mathbf{M}*\text{Exp}^{\wedge}_{G}(\boldsymbol{\epsilon}_1^{\mathbf{M}}))}{\partial\boldsymbol{\epsilon}_1^{\mathbf{M}}}^{\top}\right|_{\boldsymbol{\epsilon}_1^{\mathbf{M}}=\mathbf{0}}\left.\frac{\partial\left(\boldsymbol{\Sigma}\odot\text{Exp}^{\wedge}_{\mathcal{P}^+(s)}(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}})\right)^{-1}}{\partial\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}}\right|_{\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}=\mathbf{0}}\\ &\quad\times\left(\mathbf{z}_i-\mathbf{f}_i(\mathbf{M})\right)). \quad (48)\end{aligned}$$

By taking the expected value,

$$\begin{aligned}\mathbb{E}\left(\boldsymbol{\nabla}^2_{\mathbf{M},\boldsymbol{\Sigma}}lp(\mathbf{X},\oplus)\right) = &\left.\frac{\partial\mathbf{f}_i(\mathbf{M}*\text{Exp}^{\wedge}_{G}(\boldsymbol{\epsilon}_1^{\mathbf{M}}))}{\partial\boldsymbol{\epsilon}_1^{\mathbf{M}}}^{\top}\right|_{\boldsymbol{\epsilon}_1^{\mathbf{M}}=\mathbf{0}}\\ &\left.\frac{\partial\left(\boldsymbol{\Sigma}\odot\text{Exp}^{\wedge}_{\mathcal{P}^+(s)}(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}})\right)^{-1}}{\partial\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}}\right|_{\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}=\mathbf{0}}\mathbb{E}\left(\mathbf{z}_i-\mathbf{f}_i(\mathbf{M})\right)). \quad (49)\end{aligned}$$

As $\mathbb{E}(\mathbf{z}_i)=\mathbf{f}_i(\mathbf{M})$, the last equality is equal to $\mathbf{0}$.

Remark:

In the same way as in [27], it is worth noting that the expression of the bottom right block of the C-LG-CRB does not depend on the values of $\boldsymbol{\Sigma}$ contrary to the classical Euclidean CRB for the covariance matrix given by $\frac{2}{N}(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})^{-1}$. Thus, it can be considered as a bound that underestimates the quality of the covariance estimator. Indeed, by observing the previous expression, we remark that a covariance matrix with a certain configuration with a high matrix norm (for instance Frobenius norm), will provide a much smaller bound difficult to reach for any covariance estimator.

IV. Closed-Form Expressions

In this section, we propose to derive the C-LG-CRB for two Euclidean observation models. First, we are interested in the well-known Wahba's problem [35]. It is worth noting that the expression of the bound on LG has been already established for a known covariance matrix by using the Fisher information matrix expression based on the LG-Hessian of the log-likelihood of the observations [18]. Here, the expression is computed by using the proposed Slepian-Bangs formula on LGs (16) and C-LG-CRB. Second, we propose to deal with the pin-hole camera model which is a well-known model encountered in computer vision and camera perception [49]. Contrary to the Wahba's problem, to the best of our knowledge, for this model both LG-CRB and C-LG-CRB have not yet been computed in the state-of-the-art.

A. Wahba's problem

We assume $3D$ points $\{\mathbf{p}_i\}_{i=1}^N$ governed by the following model [35]:

$$\mathbf{z}_i = \mathbf{R}\mathbf{p}_i + \mathbf{p} + \mathbf{n}_i,\quad \mathbf{n}_i\sim\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma}), \quad (50)$$

where $\mathbf{R}$ is an unknown rotation matrix lying on $SO(3)$ and $\mathbf{p}$ an unknown translation belonging to $\mathbb{R}^3$. It can be reformulated as an observation model on $SE(3)$ with $\mathbf{M} = \begin{bmatrix}\mathbf{R} & \mathbf{p}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}$ and classical multiplicative law $*$:

$$\mathbf{z}_i = \mathbf{f}_i(\mathbf{M}) + \mathbf{n}_i \quad (51)$$

with:

$$\begin{aligned}\mathbf{f}_i(\mathbf{M}) &= \boldsymbol{\Pi}\,\mathbf{M}\tilde{\mathbf{p}}_i \quad (52)\\ \tilde{\mathbf{p}}_i &= \left[\mathbf{p}_i^{\top},1\right]^{\top} \quad (53)\\ \boldsymbol{\Pi} &= [\mathbf{I}_{3\times3},\mathbf{0}_{3\times1}]. \quad (54)\end{aligned}$$

Corollary IV-A.0.1 (Closed-form C-LG-CRB for the Wahba's problem). *The IFIM on* $\mathbf{X} = \begin{bmatrix}\mathbf{M} & \mathbf{0}_{4\times3}\\ \mathbf{0}_{3\times4} & \boldsymbol{\Sigma}\end{bmatrix}$, $\mathbf{M}\in SE(3), \boldsymbol{\Sigma}\in\mathcal{P}^+(3)$ *for the model (51) is given by*

$$\mathcal{I}_{G'} = \begin{bmatrix}\sum_{i=1}^{N}\left(\boldsymbol{\Pi}\mathbf{M}\mathbf{g}(\mathbf{p}_i)\right)^{\top}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Pi}\mathbf{M}\mathbf{g}(\mathbf{p}_i) & \mathbf{0}_{6\times6}\\ \mathbf{0}_{6\times6} & \begin{bmatrix}\frac{N}{2} & 0\\ 0 & N\end{bmatrix}\otimes\mathbf{I}_3\end{bmatrix} \quad (55)$$

with $\otimes$ *is the Kronecker product and*

$$\mathbf{g}(\mathbf{p}_i) = \begin{bmatrix}[\mathbf{p}_i]_\times & \mathbf{I}\\ 0 & \mathbf{0}_{1\times3}\end{bmatrix}. \quad (56)$$

Proof:

- First, we obtain that the column l of the Lie derivative of $\mathbf{f}_i$ is:

$$[\boldsymbol{\nabla}\mathbf{f}_i(\mathbf{M},*)]_l = \boldsymbol{\Pi}\,\mathbf{M}\,\mathbf{G}_l^{\mathfrak{se}(3)}\,\mathbf{p}_i, \tag{57}$$

where $\{\mathbf{G}_l^{\mathfrak{se}(3)}\}_{l=1}^{6}$ a basis of $\mathfrak{se}(3)$, the Lie algebra of $SE(3)$ given in Appendix A. Then, $\boldsymbol{\nabla}\mathbf{f}_i(\mathbf{M},*)$ is obtained.

- The expression of $\mathbf{B}$ given by (30) is obtained by using the natural basis of $S^{+}(3)$, with dimension $d=6$, which is given whose expressions are given in appendix (A). Then, we deduce that

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \otimes \mathbf{I}_3.$$

B. Computer vision problem

Now, we derive the C-LG-CRB for the pin-hole model. Classically, it allows to describe the pixel detection of 3D points observed by a camera. To define this model, we propose to consider a vision example where a mobile robot is equipped with a monocular camera trying to localize itself by using detection of some observed patterns, in an indoor environment.

- The mobile is characterized by its unknown pose giving the attitude $\mathbf{R} \in SO(3)$ and the position of the mobile $\mathbf{p} \in \mathbb{R}^3$,

$$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \in SE(3). \tag{58}$$

- According to the formalism proposed in [50], we assume that each pattern i is constituted of several detected 3D points. In the local frame attached to the pattern, 3D points $\{\mathbf{q}_{i,j}\}_{j=1}^{N_q}$ are assumed known and every pattern $i \in \{1,\ldots,N_P\}$ can be fully characterized by the $SE(3)$ transformation between the local pattern frame and a world frame, $\mathbf{M}_P^{(i)} = \begin{bmatrix} \mathbf{R}_P^{(i)} & \mathbf{p}_P^{(i)} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \in SE(3)$. This transformation is assumed known and can be computed in a previous camera calibration phase [51] in which camera frame and world frame are confused.

corrupted by the noise detection $\mathbf{n}_{i,j}$ which is assumed Gaussian with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}$ unknown. The Gaussian assumption is assumed valid because no physical phenomenon (for instance interference or light reflection [52], [53]), in an indoor context, has an impact on the image formation [54].

Thus, by considering $\mathbf{z}_i = \left[\mathbf{z}_{i,1}^{\top},\ldots,\mathbf{z}_{i,N_q}^{\top}\right]^{\top}$ and by noting $\pi(\mathbf{x}) \triangleq \frac{\boldsymbol{\Pi}\mathbf{x}}{[\mathbf{x}]_3}$, we have

$$\mathbf{z}_i = \mathbf{f}_i(\mathbf{M}) + \mathbf{n}_i, \tag{59}$$

with

$$\mathbf{f}_i(\mathbf{M}) = \tag{60}$$

$$\left[\mathbf{f}_{i,1}(\mathbf{M};\mathbf{M}_P^{(i)},\mathbf{q}_{i,1})^{\top},\ldots,\mathbf{f}_{i,N_q}(\mathbf{M};\mathbf{M}_P^{(i)},\mathbf{q}_{i,N_q})^{\top}\right]^{\top}, \tag{61}$$

$$\mathbf{f}_{i,j}(\mathbf{M};\mathbf{M}_P^{(i)},\mathbf{q}_{i,j}) = \mathbf{K}\pi(\mathbf{M}\mathbf{M}_p^{(i)}\left[\mathbf{q}_{i,j}^{\top},1\right]^{\top}), \tag{62}$$

$$\mathbf{n}_i = \left[\mathbf{n}_{i,1}^{\top},\ldots,\mathbf{n}_{i,N_q}^{\top}\right]^{\top}. \tag{63}$$

Note that $\mathbf{K}$ corresponds to the camera's calibration matrix.

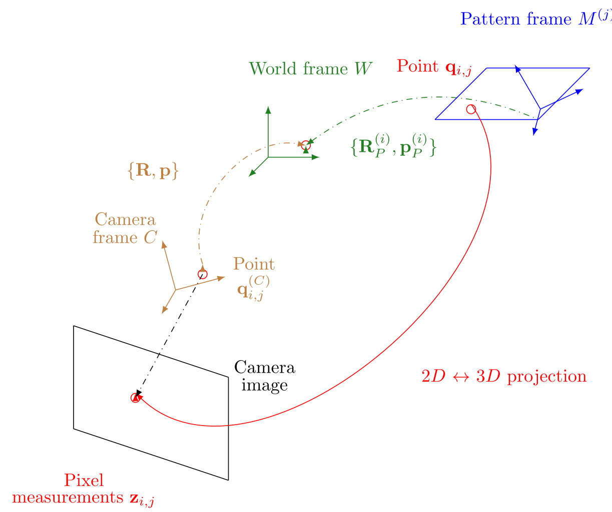


Fig. 5. Representation of geometrical transformation

Corollary IV-B.0.1 (Closed-form C-LG-CRB for the pin–hole model). *The IFIM on* $\mathbf{X} = \begin{bmatrix} \mathbf{M} & \mathbf{0}_{4\times 2} \\ \mathbf{0}_{2\times 4} & \boldsymbol{\Sigma} \end{bmatrix}, \mathbf{M} \in SE(3), \boldsymbol{\Sigma} \in \mathcal{P}^{+}(2)$ *for the model* (59) *is given by:*

$$\mathcal{I}_{G'} = \begin{bmatrix} \sum_{i=1}^{N} \boldsymbol{\nabla}\mathbf{f}_i(\mathbf{M},*)^{\top}\boldsymbol{\Sigma}^{-1}\boldsymbol{\nabla}\mathbf{f}_i(\mathbf{M},*) & \mathbf{0}_{6\times 3} \\ \mathbf{0}_{3\times 6} & \begin{bmatrix} \frac{N}{2} & 0 & 0 \\ 0 & \frac{N}{2} & 0 \\ 0 & 0 & N \end{bmatrix} \end{bmatrix}. \tag{64}$$

with:

$$N = N_P\,N_q \tag{65}$$

$$\boldsymbol{\nabla}\mathbf{f}_i(\mathbf{M},*) = \begin{bmatrix} \mathbf{D}_{i,1}^{\top} & \cdots & \mathbf{D}_{i,N_q}^{\top} \end{bmatrix}^{\top} \tag{66}$$

where $\mathbf{D}_{i,j}$ *is given by*

$$\mathbf{D}_{i,j} = \mathbf{K}\frac{\partial \pi(\mathbf{x})}{\partial \mathbf{x}}\bigg|_{\mathbf{x}=\mathbf{M}\mathbf{g}\left(\mathbf{M}_P^{(i)}[\mathbf{q}_{i,j}^{\top},1]\right)} \tag{67}$$

with $\mathbf{g}$ *defined in* (56).

Proof:

- As for the Wahba's problem, the expression of $\mathbf{D}_{i,j}$ can be achieved by using classical composition rule derivatives applied to the new function $\mathbf{f}_{i,j}(\mathbf{M})$ and use the fact that:

$$\frac{\partial \boldsymbol{\pi}(\mathbf{M}\mathrm{Exp}^{\wedge}_{SE(3)}(\boldsymbol{\delta})\,)}{\partial \boldsymbol{\delta}}\bigg|_{\boldsymbol{\delta}=\mathbf{0}} = \frac{\partial \pi(\mathbf{x})}{\partial \mathbf{x}} \tag{68}$$

computed for $\mathbf{x} = \mathbf{M}\,\frac{\partial \mathrm{Exp}^{\wedge}_{SE(3)}(\delta)}{\partial \delta}\Big|_{\boldsymbol{\delta}=\mathbf{0}}\left[\mathbf{q}_{i,j}^{\top}, 1\right]^{\top}$.

- By taking advantage of the natural basis of $S^{+}(2)$ given by:

$$\mathbf{G}_1^{\mathcal{P}} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \mathbf{G}_2^{\mathcal{P}} = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \mathbf{G}_3^{\mathcal{P}} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \tag{69}$$

$\mathbf{B}$ can be computed with the formula (30).

V. Simulations

In this section, we propose to numerically validate the proposed C-LG-CRB on the two models shown in the previous section. To achieve that, we compare it to the IMSE. In the two cases, the unknown LG parameter belongs to $G' = SE(3) \times \mathcal{P}^{+}(s)$. Then, we study the influence of the covariance structure on the behavior of the C-LG-CRB for different scenarios.

A. Details of implementation

To compute the IMSE, we use the empirical expression of its trace which can be approximated by Monte-Carlo simulation for both parameters $\mathbf{M}$ and $\boldsymbol{\Sigma}$. They are respectively given by:

$$\frac{1}{N_r}\sum_{n_r=1}^{N_r} \|\boldsymbol{l}^{*}_{SE(3)}(\mathbf{M}, \big(\widehat{\mathbf{M}}\big)_{n_r})\|^2, \tag{70}$$

$$\frac{1}{N_r}\sum_{n_r=1}^{N_r} \|\boldsymbol{l}^{\odot}_{\mathcal{P}(s)^{+}}(\mathbf{M}, \big(\widehat{\mathbf{M}}\big)_{n_r})\|^2, \tag{71}$$

where N_r is the number of realizations. The estimators $\widehat{\mathbf{M}}$ and $\widehat{\boldsymbol{\Sigma}}$ are obtained by a LG Gauss-Newton recursion, where at each iteration l, the covariance matrix is updated with its unbiased empirical estimator provided by

$$\widehat{\boldsymbol{\Sigma}}^{(l)} = \frac{1}{N-1}\sum_{i=1}^{N}\left(\mathbf{z}_i - \mathbf{f}_i(\widehat{\mathbf{M}}^{(l)})\right)\left(\mathbf{z}_i - \mathbf{f}_i(\widehat{\mathbf{M}}^{(l)})\right)^{\top}, \tag{72}$$

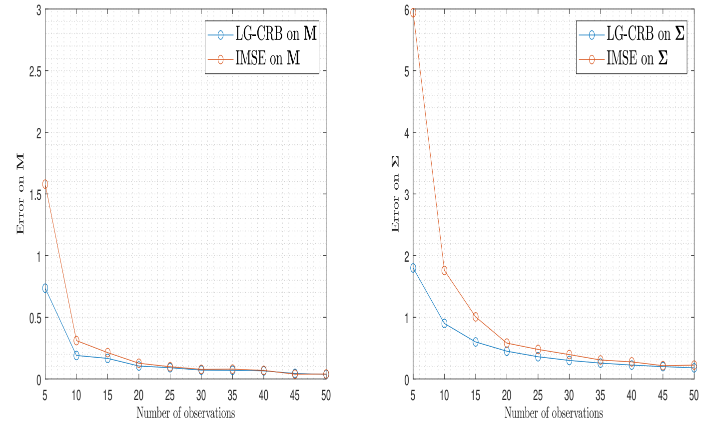
where $\widehat{\mathbf{M}}^{(l)}$ is the current estimation of $\mathbf{M}$ at iteration l. Concerning the C-LG-CRB, it is assessed by inverting the matrix $\mathcal{I}_G$. Then, the IMSE on $\mathbf{M}$ and $\boldsymbol{\Sigma}$ are respectively bounded by the trace of the top-left and bottom-right of $\mathcal{I}_G$.

B. Wahba's problem

To simulate the model, we propose to generate a number of observations, denoted N, of random points $\{\mathbf{p}_i\}_{i=1}^{N}$ in the following way:

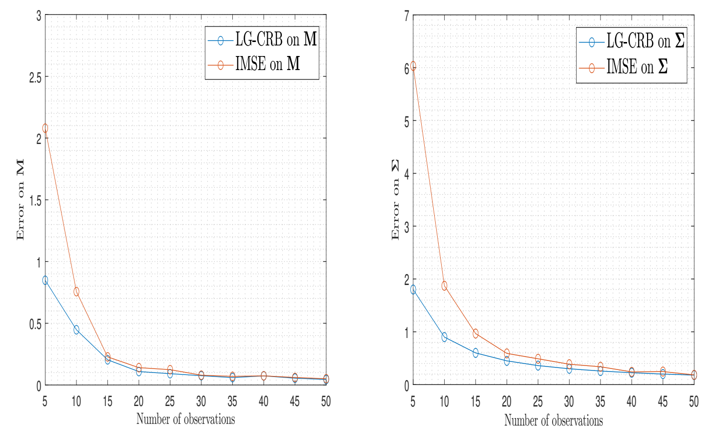
$$\mathbf{p}_i \sim \mathcal{N}_{\mathbb{R}^3}(\mathbf{p}_m, \sigma_m^2\,\mathbf{I}_3)\ \forall i \in \{1, \dots N\}, \tag{73}$$

where $\mathbf{p}_m = [1, 1, 1]^{\top}$ and $\sigma_m = 0.5$ meter. The covariance of the model is assumed to be either diagonal or non-diagonal.



(a) C-LG-CRB on $\mathbf{M}$ superimposed to the IMSE versus the number of observations. (b) C-LG-CRB on $\boldsymbol{\Sigma}$ superimposed to the IMSE versus the number of observations.

Fig. 6. Evolution of the IMSE and the C-LG-CRB for $\boldsymbol{\Sigma} = \begin{bmatrix}0.1 & 0 & 0\\ 0 & 0.1 & 0\\ 0 & 0 & 0.1\end{bmatrix}$.



(a) C-LG-CRB on $\mathbf{M}$ superimposed to the IMSE versus the number of observations. (b) C-LG-CRB on $\boldsymbol{\Sigma}$ superimposed to the IMSE versus the number of observations.

Fig. 7. Evolution of the IMSE and the C-LG-CRB for $\boldsymbol{\Sigma} = \begin{bmatrix}0.1 & 0.01 & 0.02\\ 0.01 & 0.1 & 0.01\\ 0.02 & 0.01 & 0.1\end{bmatrix}$.

By observing figures 6 and 7, we remark the consistency and convergence of the IMSE towards the C-LG-CRB for a large number of observations. When the covariance admits correlation, we observe in figures (7)-(a) and (7)-(a) that a lower number of observations induces a higher gap between IMSE and C-LG-CRB than without correlation. Nonetheless, as observed in the figures (6)-(b) and (7)-(b) we notice that the structure of $\boldsymbol{\Sigma}$ does not influence the asymptotic behavior of both IMSE and the bound. It is consistent because the covariance block of the proposed bound does not depend directly on $\boldsymbol{\Sigma}$ but only on this LG geometry. This observation can be confirmed by building several sets of correlated covariance matrices. To achieve that, we generate $\boldsymbol{\Sigma}$ with the eigenvalues-eigenvectors form $\mathbf{U}\mathbf{D}\mathbf{U}^{\top}$. $\mathbf{U} \in SO(3)$ contains the eigenvectors. We propose to parametrize $\mathbf{U}$ with the

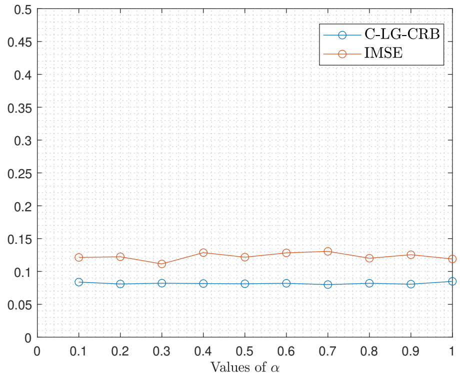


Fig. 8. Evolution of the IMSE and the C-LG-CRB versus α and $N = 100$.

following LG-form $\mathbf{U} = \text{Exp}^{\wedge}_{SO(3)}(\alpha \mathbf{1}_{3\times 1})$ where $\alpha \in \mathbb{R}$ quantifies the degree of correlation. Indeed, when $\alpha \to 0$, then $\mathbf{U} \to \mathbf{I}$. Conversely, when $\alpha \to \pi$ tends towards 0, $\mathbf{U}$ becomes non-diagonal which creates a non-diagonal structure for $\boldsymbol{\Sigma}$. In figure 8, we observe that α admits poor influence on the behavior of the pose part of the C-LG-CRB. When α becomes high i.e. the correlation on $\boldsymbol{\Sigma}$ are non-negligible, the C-LG-CRB and the IMSE stay stable. This behavior corroborates the previous observation on the fact that the correlation term does not impact the C-LG-CRB computation.

C. Pin-hole problem

Now, we are interested in simulating the pin-hole model given by the equation (59). To succeed in that, we use the pattern modeling proposed in [41]. The latter can be described by four elliptical shapess drawn where the detected points by the camera correspond to the center of each one of these shapes.

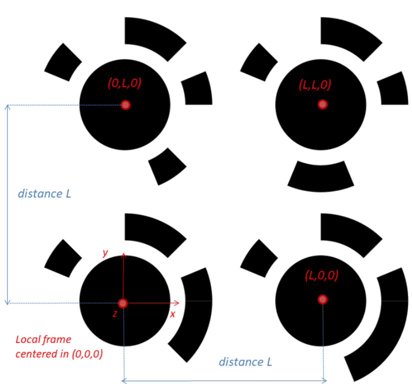
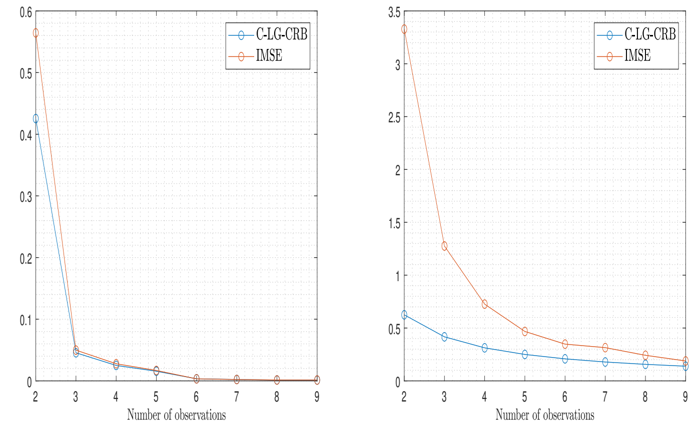


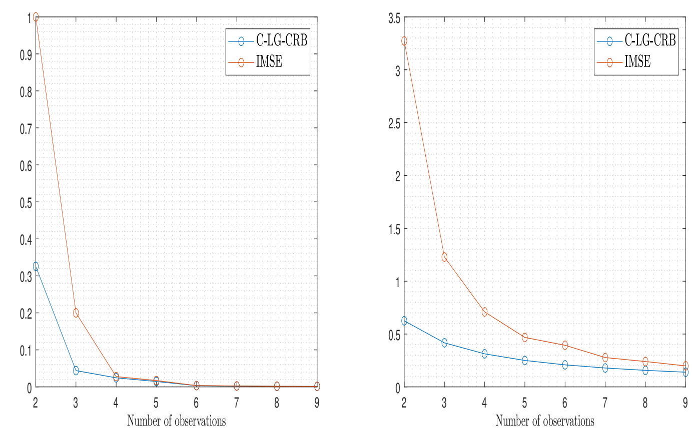
Fig. 9. Geometry of each pattern. The camera detects the center of each ellipse.

As illustrated in figure 9, the coordinates $\{\mathbf{q}_i\}_{i=1}^{4}$ of each point in the local frame attached to the pattern depend on the known inter-distance equal to L. Furthermore, we assume that the maximal number of patterns observed is equal to 9. The camera calibration matrix is fixed with standard values of ν_x and ν_y , x_o, and y_o that we can find in the literature [36]. To assess the performance of the estimator, we compute the C-LG-CRB for different values of the number of observed patterns by the camera.



(a) C-LG-CRB on $\mathbf{M}$ superimposed to the IMSE versus the number of observations (b) C-LG-CRB on $\boldsymbol{\Sigma}$ superimposed to the IMSE versus the number of observations.

Fig. 10. Evolution of the IMSE and the C-LG-CRB for $\boldsymbol{\Sigma} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$.



(a) C-LG-CRB on $\mathbf{M}$ superimposed to the IMSE versus the number of observations (b) C-LG-CRB on $\boldsymbol{\Sigma}$ superimposed to the IMSE versus the number of observations.

Fig. 11. Evolution of the IMSE and the C-LG-CRB for $\boldsymbol{\Sigma} = \begin{bmatrix} 0.1 & 0.01 \\ 0.01 & 0.1 \end{bmatrix}$.

As previously, we consider two kind of covariance matrix structures, diagonal and non-diagonal, as illustrated in figures 10 and 11. In both cases, and as in the previous model, we observe the convergence of the C-LG-CRB for the two scenarios. It is worth noting that the convergence of the IMSE for $\boldsymbol{\Sigma}$ has the same speed in the two cases. It empirically proves that the bound takes into account a full structure in the same way as a diagonal structure. This behavior is observed because the geometry is always considered. We also study more precisely the relevance of the C-LG-CRB by considering different values of the inter-distance L to observe its impact on the convergence of the IMSE to the bound. To achieve that, we plot the error between the IMSE and the C-LG-CRB as a function of L for a maximal number of patterns. This allows to quantify the convergence rate of the IMSE to the bound.

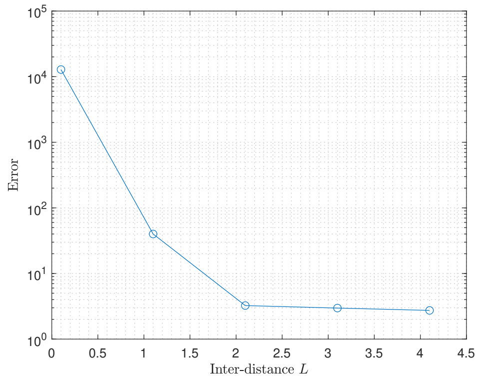


Fig. 12. Evolution of the error versus L for $N_p = 9$.

In figure 12, we observe the evolution of this error according to L. The higher L is, the more accurate the convergence rate becomes. It means that the bound can be achieved more rapidly when the camera observations are more scattered.

VI. Conclusions

In this article, we derived an intrinsic Cramér-Rao bound on Lie groups when an unknown covariance matrix needs to be considered in the estimation problem and the observations lie on an Euclidean space. The bound was established by leveraging the Lie group structure of the covariance matrix space to obtain a parameter that belongs to the product of two Lie groups. In this manner, closed-form expressions were developed and numerically validated for two models commonly used in signal processing problems. The prospects of this work are manifold. Firstly, it would be pertinent to extend the bound to cases where the observations also lie on Lie groups. Secondly, adapting it for dynamic parameters would be a challenging yet crucial endeavor, particularly in the context of tracking problems involving Lie groups, where the covariance matrix of the process model is unknown.

Appendix A
Basis Expressions

A. $\mathfrak{se}(3)$ basis

Let the vector $\mathbf{w} \in \mathbb{R}^6$ which can be divided into $[\mathbf{v}, \mathbf{u}]$. A basis of $SE(3)$ $\{\mathbf{G}_i\}_{i=1}^6$ is defined such as:

$$\begin{bmatrix} [\mathbf{v}]_\times & \mathbf{u} \\ \mathbf{0} & 0 \end{bmatrix} = \sum_{i=1}^{6} w_i \mathbf{G}_i^{\mathfrak{se}(3)} \tag{74}$$

where $[.]_\times$ is a operator transforming $\mathbf{w}$ to an anti-symmetric matrix. Then, we gather:

$$\mathbf{G}_1^{\mathfrak{se}(3)} = \begin{bmatrix} 0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix}, \mathbf{G}_2^{\mathfrak{se}(3)} = \begin{bmatrix} 0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0 \end{bmatrix}, \tag{75}$$

$$\mathbf{G}_3^{\mathfrak{se}(3)} = \begin{bmatrix} 0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \quad \mathbf{G}_4^{\mathfrak{se}(3)} = \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \tag{76}$$

$$\mathbf{G}_5^{\mathfrak{se}(3)} = \begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \quad \mathbf{G}_6^{\mathfrak{se}(3)} = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{bmatrix} \tag{77}$$

B. $S^+(3)$ basis

A natural basis of the Lie algebra $S^+(3)$ is:

$$\mathbf{G}_1^{\mathcal{P}} = \begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \mathbf{G}_2^{\mathcal{P}} = \begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}, \mathbf{G}_3^{\mathcal{P}} = \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix} \tag{78}$$

$$\mathbf{G}_4^{\mathcal{P}} = \begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix}, \mathbf{G}_5^{\mathcal{P}} = \begin{bmatrix} 0&0&1\\0&0&0\\1&0&0 \end{bmatrix}, \mathbf{G}_6^{\mathcal{P}} = \begin{bmatrix} 0&0&0\\0&0&1\\0&1&0 \end{bmatrix} \tag{79}$$

Appendix B
Demonstrations

A. Demonstration of (39)

First, we remind that for all operator $\mathbf{A} : \mathbb{R}^s \to \mathbb{R}^{s \times s}$ and $\forall \delta \in \mathbb{R}$:

$$\frac{\partial \log|\mathbf{A}(\delta)|}{\partial \delta} = \operatorname{tr}\left(\mathbf{A}(\delta)^{-1} \frac{\partial \mathbf{A}(\delta)}{\partial \delta}\right) \tag{80}$$

Furthermore:

$$\forall \mathbf{E}, \mathbf{F} \ \forall \delta \in \mathbb{R} \quad \frac{\partial \operatorname{Expm}(\mathbf{E} + \delta \mathbf{F})}{\partial \delta} = \mathbf{E} \operatorname{Expm}(\mathbf{E} + \delta \mathbf{F}) \tag{81}$$

Thus, the following expression

$$\log|\mathbf{\Sigma} \odot \operatorname{Exp}_{\mathcal{P}}^{\wedge}\left(\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}\right) \odot \operatorname{Exp}_{\mathcal{P}}^{\wedge}\left(\boldsymbol{\epsilon}_2^{\mathbf{\Sigma}}\right)| \tag{82}$$

which can be written as:

$$\log|\operatorname{Expm}\left(\operatorname{Logm}(\mathbf{\Sigma}) + sym(\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}) + sym(\boldsymbol{\epsilon}_2^{\mathbf{\Sigma}})\right)| \tag{83}$$

admits, according to (80), the following intrinsic derivative according to $\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}$:

$$\begin{aligned} &\frac{\partial}{\partial \left(\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}\right)_i} \log|\operatorname{Expm}\left(\operatorname{Logm}(\mathbf{\Sigma}) + sym(\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}) + sym(\boldsymbol{\epsilon}_2^{\mathbf{\Sigma}})\right)| \\ &= \operatorname{tr}\left(\operatorname{Expm}\left(\operatorname{Logm}(\mathbf{\Sigma}) + sym(\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}) + sym(\boldsymbol{\epsilon}_2^{\mathbf{\Sigma}})\right)^{-1} \frac{\partial \mathbf{G}(\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}, \boldsymbol{\epsilon}_2^{\mathbf{\Sigma}})}{\partial \left(\boldsymbol{\epsilon}_1^{\mathbf{\Sigma}}\right)_i}\right) \end{aligned} \tag{84}$$

with:

$$\mathbf{G}(\epsilon_1^{\boldsymbol{\Sigma}}, \epsilon_2^{\boldsymbol{\Sigma}}) = \text{Expm}\left(\text{Logm}(\boldsymbol{\Sigma}) + sym(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}) + sym(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}})\right) \tag{85}$$

Furthermore, $sym()$ can be decomposed on the natural basis of $S^+(n)$ so that:

$$sym(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}) = \sum_{i=1}^{d} \left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right)_i \mathbf{G}_i^{\mathcal{P}} \tag{86}$$

Therefore, we obtain that:

$$\frac{\partial \mathbf{B}(\epsilon_1^{\boldsymbol{\Sigma}}, \epsilon_2^{\boldsymbol{\Sigma}})}{\partial \left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right)_i} = \frac{\partial}{\partial \left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right)_i} \text{Expm}\left(\text{Logm}(\boldsymbol{\Sigma}) + \sum_{i=1}^{d} \left(\boldsymbol{\epsilon}_i^{\boldsymbol{\Sigma}}\right)_i \mathbf{G}_i^{\mathcal{P}} + sym\left(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}\right)\right) \tag{87}$$

and according to the formula (81), we have:

$$\frac{\partial \mathbf{B}(\epsilon_1^{\boldsymbol{\Sigma}}, \epsilon_2^{\boldsymbol{\Sigma}})}{\partial \left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right)_i} = \mathbf{G}_i^{\mathcal{P}} \text{Expm}\left(\text{Logm}(\boldsymbol{\Sigma}) + (\boldsymbol{\epsilon}_i)_1 \mathbf{G}_i^{\mathcal{P}} + sym\left(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}\right)\right) \tag{88}$$

$$= \mathbf{G}_i^{\mathcal{P}} \text{Expm}\left(\text{Logm}(\boldsymbol{\Sigma}) + sym\left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right) + sym\left(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}\right)\right) \tag{89}$$

Consequently, it implies that:

$$\frac{\partial}{\partial \left(\boldsymbol{\epsilon}_1\right)_i} \log|\text{Expm}\left(\text{Logm}\left(\boldsymbol{\Sigma}\right) + sym(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}) + sym(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}})\right)| = \text{tr}\left(\mathbf{G}_i^{\mathcal{P}}\right) \tag{90}$$

Then, by differentiating a new time according to $\left(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}\right)_j$, we deduce that:

$$\frac{\partial^2}{\partial \left(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}\right)_j \partial \left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right)_i} \log|\text{Expm}\left(\text{Logm}\left(\boldsymbol{\Sigma}\right) + sym(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}) + sym(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}})\right)| = 0 \tag{91}$$

B. Demonstration of (40)

By definition, we know that:

$$\left(\boldsymbol{\Sigma} \odot \text{Exp}_{\mathcal{P}}^{\wedge}\left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right) \odot \text{Exp}_{\mathcal{P}}^{\wedge}\left(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}\right)\right)^{-1}$$

$$= \text{Expm}\left(\text{Logm}(\boldsymbol{\Sigma}) + sym\left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right) + sym\left(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}\right)\right)^{-1} \tag{92}$$

$$= \text{Expm}\left(-\left(\text{Logm}(\boldsymbol{\Sigma}) + sym\left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right) + sym\left(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}\right)\right)\right) \tag{93}$$

Consequently, the derivative according to $\boldsymbol{\epsilon}_1$ can be written as:

$$\frac{\partial \text{Expm}\left(-\left(\text{Logm}(\boldsymbol{\Sigma}) + sym\left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right) + sym\left(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}\right)\right)\right)}{\partial \left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right)_i} \tag{94}$$

$$= \mathbf{G}_i^{\mathcal{P}} \text{Expm}\left(\text{Logm}(\boldsymbol{\Sigma}) + sym\left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right) + sym\left(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}\right)\right) \tag{95}$$

By differentiating a second time according to $\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}$, we have:

$$\frac{\partial \text{Expm}\left(-\left(\text{Logm}(\boldsymbol{\Sigma}) + sym\left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right) + sym\left(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}\right)\right)\right)}{\partial \left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right)_i \partial \left(\boldsymbol{\epsilon}_2^{\boldsymbol{\Sigma}}\right)_j}$$

$$= \mathbf{G}_j \mathbf{G}_i \,\text{Expm} - \left(\text{Logm}(\boldsymbol{\Sigma}) + sym\left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right) + sym\left(\boldsymbol{\epsilon}_1^{\boldsymbol{\Sigma}}\right)\right) \tag{96}$$

which provides the equation (40).

References

[1] H. L. Van Trees and K. L. Bell, Eds., *Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/Tracking*, Wiley/IEEE Press, NY, USA, 2007.

[2] H. L. Van Trees, *Detection, estimation, and modulation theory: Part I*, Wiley, New York, USA, 1968.

[3] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice-Hall, Englewood Cliffs, New Jersey, USA, 1993.

[4] P. Stoica and A. Nehorai, "Performances study of conditional and unconditional direction of arrival estimation," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 38, no. 10, pp. 1783–1795, Oct. 1990.

[5] A. Renaux, P. Forster, E. Chaumette, and P. Larzabal, "On the High SNR CML Estimator Full Statistical Characterization," *IEEE Trans. Signal Process.*, vol. 54, no. 12, pp. 4840–4843, Dec. 2006.

[6] J. Ćesić, I. Marković, M. Bukal, and I. Petrović, "Extended information filter on matrix Lie groups," *Automatica*, vol. 82, pp. 226–234, 2017.

[7] D.H.S. Maithripala and J. M. Berg, "An intrinsic PID controller for mechanical systems on Lie groups," *Automatica*, vol. 54, pp. 189–200, 2015.

[8] L. Rodrigues de Almeida, G. A. Giraldi, and M. Bernardes Vieira, "Rigid registration of point clouds based on indirect Lie group approach," in *2019 21st Symposium on Virtual and Augmented Reality (SVR)*, 2019, pp. 130–139.

[9] A. Barrau and S. Bonnabel, "Intrinsic filtering on Lie groups with applications to attitude estimation," *IEEE Transactions on Automatic Control*, vol. 60, no. 2, pp. 436–449, 2015.

[10] N. Boumal, "On Intrinsic Cramér-Rao Bounds for Riemannian Submanifolds and Quotient Manifolds," *IEEE Trans. Signal Process.*, vol. 61, no. 7, pp. 1809–1821, 2013.

[11] S. Bonnabel and A. Barrau, "An Intrinsic Cramér-Rao Bound on SO(3) for (Dynamic) Attitude Filtering," in *Proc. of the IEEE Conference on Decision and Control (CDC)*, 2015.

[12] S. Bonnabel and A. Barrau, "An Intrinsic Cramér-Rao Bound on Lie groups," in *Proc. of the International Conference on Geometric Science of Information*, 2015.

[13] G.S. Chirikjian, "Information-theoretic inequalities on unimodular Lie groups," *Journal of Geometric Mechanics*, vol. 2, pp. 119–158, 2010.

[14] S. Bonnabel and A. Barrau, "An intrinsic cramér-rao bound on SO(3) for dynamic attitude filtering," *2015 54th IEEE Conference on Decision and Control (CDC)*, pp. 2158–2163, 2015.

[15] G. S. Chirikjian, "Information theory on Lie groups and mobile robotics applications," in *IEEE International Conference on Robotics and Automation*, 2010.

[16] N. Boumal, A. Singer, P.-A. Absil, and V.D. Blondelu, "Cramér–Rao bounds for synchronization of rotations," *Information and Inference: A Journal of the IMA*, vol. 3, pp. 1–39, 2014.

[17] V. Solo and G. S. Chirikjian, "On the Cramér-Rao bound in Riemannian manifolds with application to $SO(3)$," in *IEEE International Conference on Decision and Control*, 2020.

[18] S. Labsir, A. Renaux, J. Vilà-Valls, and E. Chaumette, "Barankin, Mcaulay–seidman and Cramér–Rao bounds on matrix Lie groups," *Automatica*, vol. 156, pp. 111199, 2023.

[19] E. W. Barankin, "Locally best unbiased estimates," *Annals of Mathematical Statistics*, vol. 20, pp. 477–501, 1949.

[20] D. Medina, J. Vilà-Valls, E. Chaumette, F. Vincent, and P. Closas, "Cramér-Rao Bound for a Mixture of Real- and Integer-Valued Parameter Vectors and Its Application to the Linear Regression Model," *Signal Processing*, vol. 179, pp. 107792, Feb. 2021.

[21] R. McAulay and L. P. Seidman, "A useful form of the Barankin lower bound and its application to PPM threshold analysis," *IEEE Trans. Inf. Theory*, vol. 15, no. 2, pp. 273–279, Mar. 1969.

[22] J. W. Koch, "Bayesian approach to extended object and cluster tracking using random matrices," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 44, no. 3, pp. 1042–1059, 2008.

[23] H. Alqaderi, F. Govaers, and W. Koch, "Bayesian Wishart filter for random shape tracking," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 58, no. 3, pp. 1941–1952, 2022.

[24] B. Goossens, H. Luong, J. Aelterman, A. Pizurica, and W. Philips, "Realistic camera noise modeling with application to improved hdr synthesis," *EURASIP Journal on Advances in Signal Processing*, vol. 2012, 08 2012.

[25] Rihuan K., "Deep variation prior: Joint image denoising and noise variance estimation without clean data," 2022.

[26] S. T. Smith, "Covariance, Subspace, and Intrinsic Cramér-Rao Bounds," *IEEE Trans. Signal Process.*, vol. 53, no. 5, pp. 1610–1630, 2005.

[27] A. Breloy, G. Ginolhac, A. Renaux, and F. Bouchard, "Intrinsic Cramér-Rao Bounds for Scatter and Shape Matrices Estimation in CES Distributions," *IEEE Sig. Proc. Letters*, vol. 26, no. 2, pp. 262–266, 2019.

[28] F. Bouchard, A. Breloy, A. Renaux, and G. Ginolhac, "Riemannian geometry and Cramér-Rao bound for blind separation of gaussian sources," in *ICASSP 2020 - 2020 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2020, pp. 4717–4721.

[29] S. Labsir, A. Giremus, B. Yver, and T. Benoudiba-Campanini, "Joint shape and centroid position tracking of a cluster of space debris by filtering on Lie groups," *Elsevier Signal Processing*, vol. 183, pp. 108027, 2021.

[30] S. Labsir, A. Giremus, B. Yver, and T. Benoudiba–Campanini, "A Lie-group based modelling for centroid and shape estimation of a cluster of space debris," in *2020 28th European Signal Processing Conference (EUSIPCO)*, 2021, pp. 960–964.

[31] V. Arsigny, P. Fillard, X. Pennec, and N. Ayache, "Geometric means in a novel vector space structure on symmetric positive-definite matrices," *SIAM Journal on Matrix Analysis and Applications*, vol. 29, no. 1, pp. 328–347, 2007.

[32] I. Ilea, L. Bombrun, S. Said, and Y. Berthoumieu, "Covariance matrices encoding based on the log-euclidean and affine invariant riemannian metrics," in *2018 IEEE/CVF Conference on Computer Vision and Pattern Recognition Workshops (CVPRW)*, 2018, pp. 506–50609.

[33] J. Gallier and J. Quaintance, *The Log-Euclidean Framework Applied to SPD Matrices*, pp. 665–673, Springer International Publishing, Cham, 2020.

[34] Zhiwu H., Ruiping W., S. Shan, Xianqiu L., and Xilin C., "Log-Euclidean metric learning on symmetric positive definite manifold with application to image set classification," in *International Conference on Machine Learning*, 2015.

[35] G. Wahba, "A least squares estimate of satellite attitude," *SIAM Review*, vol. 7, no. 3, pp. 409–409, 1965.

[36] K.K. Tseng, Jun Li, Yachin Chang, K.L. Yung, and Andrew W.H. Ip, "Chapter 9 - monocular simultaneous localization and mapping for a space rover application," in *IoT and Spacecraft Informatics*, K.L. Yung, Andrew W.H. Ip, Fatos Xhafa, and K.K. Tseng, Eds., Aerospace Engineering, pp. 285–305. Elsevier, 2022.

[37] J.M. Selig, *Geometrical methods in robotics*, Springer, 1996.

[38] J. Faraut, "Analysis on Lie groups : an introduction," chapter 5, p. 75. Cambridge, 2008.

[39] Gregory S Chirikjian, *Stochastic models, information theory, and Lie groups, volume 2: Analytic methods and modern applications*, vol. 2, Springer Science & Business Media, 2011.

[40] N. Boumal, *Optimization and estimation on manifolds*, Ph.D. thesis, Université catholique de Louvain, feb 2014.

[41] S. Labsir, A. Giremus, B. Yver, and T. Benoudiba-Campanini, "Joint Shape and Centroid Position Tracking of a Cluster of Space Debris by Filtering on Lie groups," *Signal Process.*, vol. 183, pp. 108027, 2021.

[42] G. Bourmaud, R. Mégret, A. Giremus, and Y. Berthoumieu, "From Intrinsic Optimization to Iterated Extended Kalman Filtering on Lie groups," *Journal of Mathematical Imaging and Vision*, vol. 55, no. 3, July 2016.

[43] G. Bourmaud, "Online Variational Bayesian Motion Averaging," in *ECCV*, Amsterdam, Netherlands, 2016.

[44] S. Bonnabel and A. Barrau, "An intrinsic Cramér-Rao bound on Lie groups," in *International Conference on Networked Geometric Science of Information*, 2015, pp. 664–672.

[45] George AF Seber, *A matrix handbook for statisticians*, John Wiley & Sons, 2008.

[46] S.T. Smith, "Covariance, subspace, and intrinsic crame/spl acute/r-rao bounds," *IEEE Transactions on Signal Processing*, vol. 53, no. 5, pp. 1610–1630, 2005.

[47] Slepian D., "Estimation of signal parameters in the presence of noise,," *Trans. IRE Prof. Group Inform. Theory PG IT-3*, vol. 74, pp. 68–69, 1954.

[48] W. J. Bangs, *Array processing with generalized beamformers*, Ph.D. thesis, Yale Univ., New Haven, CT,, Novembre 1971.

[49] P. Sturm, *Pinhole Camera Model*, Springer US, 2014.

[50] S. Labsir, G. Pages, and D. Vivet, "Lie group modelling for an EKF-based monocular slam algorithm," *Remote Sensing*, vol. 14, no. 3, 2022.

[51] Andréa Macario Barros, Maugan Michel, Yoann Moline, Gwenolé Corre, and Frédérick Carrel, "A comprehensive survey of visual SLAM algorithms," *Robotics*, vol. 11, no. 1, 2022.

[52] Pradeep Ranganathan and Edwin Olson, "Gaussian process for lens distortion modeling," in *2012 IEEE/RSJ International Conference on Intelligent Robots and Systems*, 2012, pp. 3620–3625.

[53] Jeffrey A. Fessler, "Spatial resolution and noise tradeoffs in pinhole imaging system design: a density estimation approach," *Opt. Express*, vol. 2, no. 6, pp. 237–253, Mar 1998.

[54] Otto Korkalo and Tapio Takala, "Measurement noise model for depth camera-based people tracking," *Sensors*, vol. 21, no. 13, 2021.