Delay estimation with a carrier modulated by a band-limited signal

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***Abstract*—Since time-delay estimation is a fundamental task in various engineering fields, several expressions for the CRB and MLE have been developed over the past decades. In all of these previous studies, a common assumption was that the wave transmission process introduced an unknown phase component, which made it impossible to exploit the phase component related to the delay from the carrier signal. However, there are practical scenarios where this unknown phase can be estimated and compensated for, enabling the utilization of the delay phase component from the carrier signal. In this context, we provide a comprehensive treatment of this scenario, including the derivation of the MLE and the associated CRB. This approach allows us to analyze the impact of each signal component (carrier frequency and baseband signal) on the achievable MSE of delay estimation relative to the SNR. It also reveals five distinct regions of operations, in contrast to the well-known three.**

***Index Terms*—Cramér-Rao bound, maximum likelihood estimation, time-delay, band-limited signals.**

I. Introduction

Time-delay estimation is a critical task in numerous engineering fields, including navigation, radar, reflectometry, sonar, and communications, among others [1]–[9]. Estimating this parameter serves as a crucial initial step in the receiver's operation [5], [8], [9]. In the design and evaluation of estimation techniques for these applications, it is imperative to understand the ultimate achievable performance in terms of Mean Squared Error (MSE). This valuable information can be provided by Cramér-Rao bounds (CRB) [10], the most widely used lower bound on MSE due to its ease of calculation for various problems (see [4, §8.4] and [11, Part III]). Furthermore, the CRB accurately estimates the MSE of the Maximum Likelihood Estimator (MLE) in the asymptotic region of operation under certain conditions, such as the large sample regime and/or high Signal-to-Noise Ratio (SNR) regime of the Gaussian conditional signal model (CSM) [12], [13]. As a result, it is not surprising that several CRB expressions have been derived for the delay-Doppler estimation problem over the past few decades. These expressions cover a range of signal types, including finite narrow-band signals [2], [14]–[22], finite wide-band signals [14], [17], [20], [23]–[26], and infinite bandwidth signals [27]. In all of these prior studies, a common assumption is made: the wave transmission process introduces an unknown phase component, which hinders the utilization of the time-delay term associated to the carrier-phase. This unknown phase component stems from the assumption of imperfect knowledge of the propagation medium, of the transmitter and/or receiver antenna characteristics (including center of phase and hyper-frequency electronics) and/or the radio-frequency electronics. In these studies, all phase components are aggregated and combined with the amplitude component, resulting in an unknown complex amplitude. This perhaps explains why a comprehensive treatment of scenarios in which this unknown phase component can be estimated and compensated for seems to be lacking in the existing literature. However, a case of interest is, for instance, a ground-based navigation scenario where the transmitter remains static, and the receiver plans to move from one known location to another (way-points). At each known location, the receiver can estimate the phase component attributable to the transmission process [3] and compensate for it as it travels towards the next known location. This would enable the leverage of the carrier-phase term for a finer time-delay estimation performance assessment, since the resulting signal model would characterize better the received signal in the receiver's end. To simplify matters, it is first considered the case where the Doppler effect, mainly a carrier frequency shift in general, can be neglected (e.g., in the case of a static or slowly moving transmitter and receiver). Given the specific CSM in such scenarios, this communication presents a novel MLE and CRB, and extends the known relationship for the standard CSM between the CRB and the ambiguity function [2, § 10] [5, § 3.9.4]. The results presented herein allow us to analyze the impact of each signal component (carrier frequency and baseband signal) on the achievable MSE of time-delay estimation concerning SNR, revealing five regions of operation instead of the commonly known three [28], [29].

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II. Signal Model and MLE

We consider the transmitter-to-receiver direct transmission of a band-limited signal $a(t)$ with bandwidth B

$$a(t) = \sum_{n=N_1}^{N_2} a(nT)\,\mathrm{sinc}\,(\pi B(t-nT)),\ T = 1/B, \quad (1)$$

over a carrier with frequency f_c. If the transmitter to receiver distance is constant (constant propagation delay τ) during observations, the baseband signal at the output of the receiver's Hilbert filter with bandwidth $F_s \geq B$ can be modelled as [2], [12], [21], [30]:

$$x(t) = \alpha e^{j\phi} a(t-\tau) + n(t),\ \phi = \psi + \varphi(\tau), \quad (2)$$

where $\alpha e^{j\psi}$ is the transmission loss complex amplitude, $\alpha \in \mathbb{R}$ and $\alpha > 0$, $\varphi(\tau) = -2\pi f_c \tau$ for constant transmitter to receiver distance, and $n(t)$ is a complex centered circular Gaussian noise, white within the bandwidth F_s with variance σ_n^2. If ψ can be measured, i.e., calibrated, and compensated for, then (2) becomes

$$x(t) = \alpha e^{j\varphi(\tau)} a(t-\tau) + n(t),\ \varphi(\tau) = -2\pi f_c \tau,\ \alpha > 0. \quad (3)$$

The discrete vector signal model is build from $N' = N_2' - N_1' + 1$ ($N_1' \ll N_1$, $N_2' \gg N_2$) samples at $T_s = 1/F_s$,

$$\mathbf{x} = \alpha e^{j\varphi(\tau)} \mathbf{a}(\tau) + \mathbf{n}, \quad (4)$$

with signal samples $\mathbf{x} = [x(N_1'T_s), \ldots, x(N_2'T_s)]^\top$, noise samples $\mathbf{n} = [n(N_1'T_s), \ldots, n(N_2'T_s)]^\top$, code samples $\mathbf{a}(\tau) = [a(N_1'T_s - \tau), \ldots, a(N_2'T_s - \tau)]^\top$. Actually, signal model (4) is a particular instantiation of the general CSM [12], [31]

$$\mathbf{x} = \mathbf{s}(\zeta) + \mathbf{n},\ \mathbf{s}(\zeta) = \mathbf{a}'(\tau)\alpha,\ \zeta = (\alpha, \tau)^T,\ \alpha > 0, \quad (5)$$

where $\mathbf{a}'(\tau) = \mathbf{a}(\tau) e^{j\varphi(\tau)}$ and $\varphi(\tau) = -j2\pi f_c \tau$, the unknown deterministic parameter vector to estimate being $\epsilon = (\sigma_n^2, \tau, \alpha)$. It is noteworthy that the MLE of $\{\tau, \alpha\}$ in (5) is partly constrained

$$(\widehat{\tau}, \widehat{\alpha}) = \arg\min_{\tau, \alpha > 0} \left\{ \left\| \mathbf{x} - \mathbf{a}'(\tau)\alpha \right\|^2 \right\}, \quad (6)$$

which is a different setting than the one usually assumed in the CSM [12], [31], and for the MLE with vector parameter equality constaints [32]. In the following, let $\epsilon^0 = \left(\left(\sigma_n^2\right)^0, \left(\zeta^0\right)^T \right)^T$, $\zeta^0 = \left(\tau^0, \alpha^0\right)^T$, be the selected (true) value of ϵ, i.e.,

$$\mathbf{x} = \mathbf{a}\left(\tau^0\right) e^{j\varphi(\tau^0)} \alpha^0 + \mathbf{n},\ \mathbf{n} \sim \mathcal{CN}\left(0, \left(\sigma_n^2\right)^0 \mathbf{I}_N\right), \quad (7)$$

and let $\underline{\mathbf{a}}'(\tau)^T = \left(\Re\{\mathbf{a}'(\tau)\}^T, \Im\{\mathbf{a}'(\tau)\}^T \right)$, $\underline{\mathbf{x}}^T = \left(\Re\{\mathbf{x}\}^T, \Im\{\mathbf{x}\}^T \right)$ and $\underline{\mathbf{n}}^T = \left(\Re\{\mathbf{n}\}^T, \Im\{\mathbf{n}\}^T \right) \sim \mathcal{N}\left(\mathbf{0}, (\sigma_n^2/2) \cdot \mathbf{I}_{2N}\right)$, such that

$$\mathbf{x} = \mathbf{a}'(\tau)\alpha + \mathbf{n} \Leftrightarrow \underline{\mathbf{x}} = \underline{\mathbf{a}}'(\tau)\alpha + \underline{\mathbf{n}}. \quad (8)$$

A. MLE with inequality constraint

Since[1]

$$\underline{\mathbf{a}}'(\tau)^T \underline{\mathbf{a}}'(\tau) = \mathbf{a}'(\tau)^H \mathbf{a}'(\tau),\ \underline{\mathbf{a}}'(\tau)^T \underline{\mathbf{x}} = \Re\left\{ \mathbf{a}'(\tau)^H \mathbf{x} \right\},$$

$$\left\| \Pi_{\underline{\mathbf{a}}'(\tau)} \underline{\mathbf{x}} \right\|^2 = \underline{\mathbf{x}}^T \frac{\underline{\mathbf{a}}'(\tau) \underline{\mathbf{a}}'(\tau)^T}{\underline{\mathbf{a}}'(\tau)^T \underline{\mathbf{a}}'(\tau)} \underline{\mathbf{x}} = \Re\left\{ \frac{\mathbf{a}'(\tau)^H}{\|\mathbf{a}'(\tau)\|} \mathbf{x} \right\}^2, \quad (9a)$$

and

$$\left\| \mathbf{x} - \mathbf{a}'(\tau)\alpha \right\|^2 = \left\| \underline{\mathbf{x}} - \underline{\mathbf{a}}'(\tau)\alpha \right\|^2, \quad (9b)$$

then

$$\begin{aligned} &\left\| \mathbf{x} - \mathbf{a}'(\tau)\alpha \right\|^2 = \left\| \underline{\mathbf{x}} - \underline{\mathbf{a}}'(\tau)\alpha \right\|^2 \\ &= \left\| \Pi_{\underline{\mathbf{a}}'(\tau)} \left(\underline{\mathbf{x}} - \underline{\mathbf{a}}'(\tau)\alpha \right) \right\|^2 + \left\| \Pi^\perp_{\underline{\mathbf{a}}'(\tau)} \left(\underline{\mathbf{x}} - \underline{\mathbf{a}}'(\tau)\alpha \right) \right\|^2 \\ &= \left\| \underline{\mathbf{a}}'(\tau) \right\|^2 \left(\widehat{\alpha}_u(\tau) - \alpha \right)^2 + \left\| \Pi^\perp_{\underline{\mathbf{a}}'(\tau)} \underline{\mathbf{x}} \right\|^2, \end{aligned}$$

where

$$\widehat{\alpha}_u(\tau) = \frac{\underline{\mathbf{a}}'(\tau)^T \underline{\mathbf{x}}}{\underline{\mathbf{a}}'(\tau)^T \underline{\mathbf{a}}'(\tau)} = \frac{\Re\left\{ \mathbf{a}'(\tau)^H \mathbf{x} \right\}}{\mathbf{a}'(\tau)^H \mathbf{a}'(\tau)},$$

denotes the usual unconstrained estimator [31] (that is, the estimator of α if the constraint $\alpha > 0$ is relaxed). Hence,

$$\begin{aligned} &(\widehat{\tau}, \widehat{\alpha}) = \arg\min_{\tau, \alpha > 0} \left\{ \left(\widehat{\alpha}_u(\tau) - \alpha \right)^2 \left\| \underline{\mathbf{a}}'(\tau) \right\|^2 + \left\| \Pi^\perp_{\underline{\mathbf{a}}'(\tau)} \underline{\mathbf{x}} \right\|^2 \right\}, \\ &\Rightarrow \forall \tau : \widehat{\alpha} = \arg\min_{\alpha > 0} \left\{ \left(\widehat{\alpha}_u(\tau) - \alpha \right)^2 \right\}. \quad (10) \end{aligned}$$

Thus

- if $\widehat{\alpha}_u(\tau) > 0$ then: $\min_{\alpha > 0} \left\{ \left(\alpha - \widehat{\alpha}_u(\tau) \right)^2 \right\} = 0$ and $\widehat{\alpha}(\tau) = \widehat{\alpha}_u(\tau)$,
- if $\widehat{\alpha}_u(\tau) \leq 0$ then: $\min_{\alpha > 0} \left\{ \left(\alpha - \widehat{\alpha}_u(\tau) \right)^2 \right\} = \widehat{\alpha}_u^2(\tau)$ and $\widehat{\alpha}(\tau) = 0$,

which yields to

$$\left\| \mathbf{x} - \mathbf{a}'(\tau)\widehat{\alpha} \right\|^2 = \|\mathbf{x}\|^2 - \Re\left\{ \frac{\mathbf{a}'(\tau)^H \mathbf{x}}{\|\mathbf{a}'(\tau)\|} \right\}^2 1_{\Re\{\mathbf{a}'(\tau)^H \mathbf{x}\} > 0}, \quad (11)$$

where $1_{\mathcal{D}}$ is the indicator function of subset $\mathcal{D}$ of $\mathbb{R}$. Finally

$$\widehat{\tau} = \arg\min_{\{\tau | \Re\{\mathbf{a}'(\tau)^H \mathbf{x}\} > 0\}} \left\{ \|\mathbf{x}\|^2 - \Re\left\{ \frac{\mathbf{a}'(\tau)^H \mathbf{x}}{\|\mathbf{a}'(\tau)\|} \right\}^2 \right\}, \quad (12a)$$

or equivalently,

$$\widehat{\tau} = \arg\max_{\{\tau | \Re\{\mathbf{a}'(\tau)^H \mathbf{x}\} > 0\}} \left\{ \Re\left\{ \frac{\mathbf{a}'(\tau)^H \mathbf{x}}{\|\mathbf{a}'(\tau)\|} \right\}^2 \right\}. \quad (12b)$$

[1] Let $S = span(\mathbf{A})$, with $\mathbf{A}$ a matrix, be the linear span of the set of its column vectors, and $S^\perp$ the orthogonal complement of the subspace S. The orthogonal projector over S and $S^\perp$ are $\mathbf{\Pi_A} = \mathbf{A}\left(\mathbf{A}^H \mathbf{A}\right) \mathbf{A}^H$ and $\mathbf{\Pi}^\perp_{\mathbf{A}} = \mathbf{I} - \mathbf{\Pi_A}$.

In the case of (4), (12b) becomes

$$\widehat{\tau} = \arg \max_{\left\{\tau \mid \Re\left\{e^{-j\varphi(\tau)}\left(\mathbf{a}(\tau)^H \mathbf{x}\right)\right\}>0\right\}} \left\{ \Re\left\{ e^{-j\varphi(\tau)} \frac{\mathbf{a}^H(\tau)\mathbf{x}}{\|\mathbf{a}(\tau)\|} \right\}^2 \right\}. \tag{13}$$

III. CRB FOR BAND-LIMITED SIGNALS

As shown in [33], in case of a parameter constraint, the CRB is unchanged at a regular point, i.e. where no equality constraint is active. Thus for $\alpha > 0$, the CRB is obtained from the standard Fisher Information Matrix (FIM) $\mathbf{F}\left(\epsilon^0\right)$,

$$\mathbf{CRB}_{\epsilon|\epsilon}\left(\epsilon^0\right) = \mathbf{F}\left(\epsilon^0\right)^{-1}, \tag{14a}$$

derived from the Slepian-Bangs formula [10, (3.31)] where $\underline{\mathbf{x}} \sim \mathcal{N}\left(\mathbf{m}_{\underline{\mathbf{x}}}(\epsilon), \mathbf{C}_{\underline{\mathbf{x}}}(\epsilon)\right)$, $\epsilon^T = \left(\sigma_n^2, \alpha, \tau\right)$, $\mathbf{m}_{\underline{\mathbf{x}}}(\epsilon) = \underline{\mathbf{a}}'(\tau)\alpha$, $\mathbf{C}_{\underline{\mathbf{x}}}(\epsilon) = (\sigma_n^2/2)\mathbf{I}_{2N}$. Then

$$\mathbf{F}(\epsilon) = \begin{bmatrix} \frac{1}{2} tr\left(\left(\frac{\sigma_n^2}{2}\right)^{-2} \frac{1}{2}\mathbf{I}_{2N}\right) & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{F}(\zeta) \end{bmatrix}, \tag{14b}$$

$$\mathbf{F}(\zeta) = \frac{2}{\sigma_n^2} \begin{bmatrix} \underline{\mathbf{a}}'(\tau)^T \underline{\mathbf{a}}'(\tau) & \alpha \frac{\partial \underline{\mathbf{a}}'(\tau)}{\partial \tau}^T \underline{\mathbf{a}}'(\tau) \\ \alpha \frac{\partial \underline{\mathbf{a}}'(\tau)}{\partial \tau}^T \underline{\mathbf{a}}'(\tau) & \alpha^2 \frac{\partial \underline{\mathbf{a}}'(\tau)}{\partial \tau}^T \frac{\partial \underline{\mathbf{a}}'(\tau)}{\partial \tau} \end{bmatrix}, \tag{14c}$$

leading to

$$CRB_{\tau|\epsilon}(\epsilon) = \frac{\sigma_n^2}{2\alpha^2}\Phi_r^{-1}, \tag{15a}$$

$$\Phi_r = \frac{\partial \underline{\mathbf{a}}'(\tau)}{\partial \tau}^T \mathbf{\Pi}_{\underline{\mathbf{a}}'(\tau)}^{\perp} \frac{\partial \underline{\mathbf{a}}'(\tau)}{\partial \tau} = \left\| \frac{\partial \mathbf{a}'(\tau)}{\partial \tau} \right\|^2 - \frac{\Re\left\{ \mathbf{a}'(\tau)^H \frac{\partial \mathbf{a}'(\tau)}{\partial \tau} \right\}^2}{\|\mathbf{a}'(\tau)\|^2}. \tag{15b}$$

More specifically, if $\mathbf{a}'(\tau) = \mathbf{a}(\tau) e^{j\varphi(\tau)}$, then

$$\Phi_r = \left\| \frac{\partial \mathbf{a}(\tau)}{\partial \tau} \right\|^2 - \frac{\Re\left\{ \mathbf{a}(\tau)^H \frac{\partial \mathbf{a}(\tau)}{\partial \tau} \right\}^2}{\|\mathbf{a}(\tau)\|^2} + \left(\frac{\partial \varphi(\tau)}{\partial \tau} \right)^2 \|\mathbf{a}(\tau)\|^2 - 2\frac{\partial \varphi(\tau)}{\partial \tau} \Im\left\{ \mathbf{a}(\tau)^H \frac{\partial \mathbf{a}(\tau)}{\partial \tau} \right\}. \tag{15c}$$

Moreover, according to the Nyquist-Shannon theorem,

$$\lim_{(N_1', N_2') \to (-\infty, +\infty)} \Phi_r = F_s \left(w_2 - \frac{\Re\{w_3\}^2}{w_1} + 4\pi^2 f_c^2 w_1 + 4\pi f_c \Im\{w_3\} \right),$$

where $w_1 = \int_{-\infty}^{+\infty} |a(t)|^2 dt = \frac{\mathbf{a}^H \mathbf{a}}{F_s}$, $w_2 = \int_{-\infty}^{+\infty} \left| \frac{\partial a(t)}{\partial \tau} \right|^2 dt = F_s \mathbf{a}^H \mathbf{V} \mathbf{a}$, and $w_3 = \int_{-\infty}^{+\infty} \frac{\partial a(t)}{\partial \tau} a(t)^* dt = \mathbf{a}^H \mathbf{\Lambda} \mathbf{a}$, with $\mathbf{V}$ a symmetric positive definite real-valued matrix and $\mathbf{\Lambda}$ an anti-symmetric real-valued matrix detailled in [24]. Finally Φ_r for $CRB_{\tau|\epsilon}(\epsilon)$ (15a) can be recasted as

$$\Phi_r = F_s^2 \mathbf{a}^H \mathbf{a} \left(\begin{array}{c} \frac{\mathbf{a}^H \mathbf{V} \mathbf{a}}{\mathbf{a}^H \mathbf{a}} - \Re\left\{ \frac{\mathbf{a}^H \mathbf{\Lambda} \mathbf{a}}{\mathbf{a}^H \mathbf{a}} \right\}^2 + \\ 4\pi^2 \left(\frac{f_c}{F_s} \right)^2 + 4\pi \frac{f_c}{F_s} \Im\left\{ \frac{\mathbf{a}^H \mathbf{\Lambda} \mathbf{a}}{\mathbf{a}^H \mathbf{a}} \right\} \end{array} \right), \tag{15d}$$

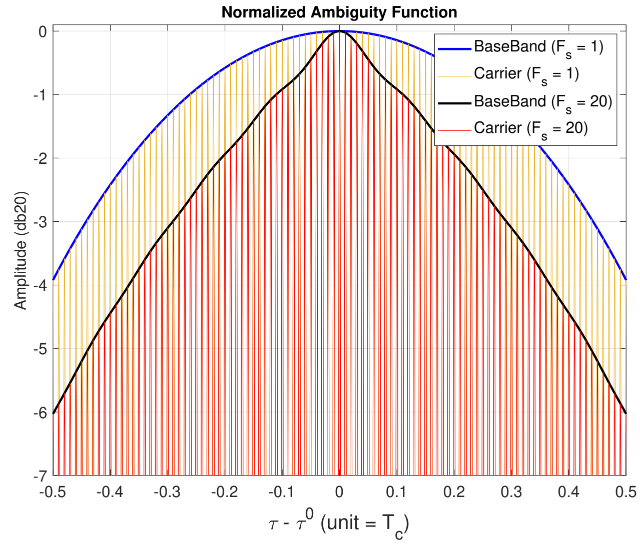

Fig. 1. Normalized ambiguity function for carrier signal $\Xi\left(\tau; \tau^0\right)$ (17) and baseband signal $\Xi_a\left(\tau; \tau^0\right)$ (20). $F_s \in \{1, 20\}$, $f_c = 100F_c$, $a(t)$ is 1 Gold code sequence (1023 bits).

an expression which depends only on the carrier f_c and the baseband signal samples $\mathbf{a}$. Note that if $f_c \gg F_s$, then Φ_r reduces to $\Phi_r \simeq \left(2\pi \|\mathbf{a}\| f_c\right)^2$ and the $CRB_{\tau|\epsilon}(\epsilon)$ depends only on the energy carried by the baseband signal and no longer on its temporal of frequential shape.

IV. AMBIGUITY FUNCTION AND SNR THRESHOLD

From (9a) and (12b),

$$\widehat{\tau} = \arg \max_{\left\{\tau \mid \underline{\mathbf{a}}'(\tau)^T \underline{\mathbf{x}} > 0\right\}} \left\{ \left\| \Pi_{\underline{\mathbf{a}}'(\tau)} \underline{\mathbf{x}} \right\|^2 \right\}, \quad \underline{\mathbf{x}} = \underline{\mathbf{a}}'\left(\tau^0\right)\alpha^0 + \underline{\mathbf{n}}.$$

Thus the MLE score function reads

$$\left\| \Pi_{\underline{\mathbf{a}}'(\tau)} \underline{\mathbf{x}} \right\|^2 = \left\| \Pi_{\underline{\mathbf{a}}'(\tau)} \underline{\mathbf{a}}'\left(\tau^0\right)\alpha^0 + \Pi_{\underline{\mathbf{a}}'(\tau)} \underline{\mathbf{n}} \right\|^2, \tag{16}$$

which at high SNR tends to the so-called "ambiguity function" [30]

$$\Xi\left(\tau; \tau^0\right) = \left\| \Pi_{\underline{\mathbf{a}}'(\tau)} \underline{\mathbf{a}}'\left(\tau^0\right)\alpha^0 \right\|^2 = \left(\alpha^0\right)^2 \frac{\left(\underline{\mathbf{a}}'(\tau)^T \underline{\mathbf{a}}'\left(\tau^0\right)\right)^2}{\underline{\mathbf{a}}'(\tau)^T \underline{\mathbf{a}}'(\tau)} = \left(\alpha^0\right)^2 \frac{\Re\left\{ \mathbf{a}'(\tau)^H \mathbf{a}'\left(\tau^0\right) \right\}^2}{\mathbf{a}'(\tau)^H \mathbf{a}'(\tau)}. \tag{17}$$

A few lines of calculus (not detailed here) allow to prove that

$$\Xi\left(\tau^0 + d\tau; \tau^0\right) \simeq \Xi\left(\tau^0; \tau^0\right) + \frac{1}{2} \left. \frac{\partial^2 \Xi\left(\tau; \tau^0\right)}{\partial^2 \tau} \right|_{\tau^0} d\tau^2 \simeq \left(\alpha^0\right)^2 \left\| \mathbf{a}'\left(\tau^0\right) \right\|^2 \left(1 - \frac{1}{2} \left(\frac{2\Phi_r}{\left\| \mathbf{a}'\left(\tau^0\right) \right\|^2} \right) d\tau^2 \right) \tag{18a}$$

and according to (15a),

$$CRB_{\tau|\epsilon}\left(\epsilon^0\right) = F(\tau)^{-1}, \quad F(\tau) = \frac{-1}{\sigma_n^2} \left. \frac{\partial^2 \Xi\left(\tau; \tau^0\right)}{\partial^2 \tau} \right|_{\tau^0}, \tag{18b}$$

which generalizes the result known for the standard CSM [2, § 10] [5, § 3.9.4]. In the case of (4), (18a) becomes

$$\Xi\left(\tau^0+d\tau;\tau^0\right)=\left(\alpha^0\right)^2\left\|\mathbf{a}\left(\tau^0\right)\right\|^2\times \\ \Re\left\{\frac{\mathbf{a}\left(\tau^0+d\tau\right)^H\mathbf{a}\left(\tau^0\right)}{\left\|\mathbf{a}\left(\tau^0+d\tau\right)\right\|\left\|\mathbf{a}\left(\tau^0\right)\right\|}e^{j\left(\varphi\left(\tau^0\right)-\varphi\left(\tau^0+d\tau\right)\right)}\right\}^2. \quad (19)$$

At the vicinity of τ^0, since

$$\frac{\mathbf{a}\left(\tau^0+d\tau\right)^H\mathbf{a}\left(\tau^0\right)}{\left\|\mathbf{a}\left(\tau^0+d\tau\right)\right\|\left\|\mathbf{a}\left(\tau^0\right)\right\|}\in\mathbb{R},$$

$$\varphi\left(\tau^0\right)-\varphi\left(\tau^0+d\tau\right)\simeq-\frac{\partial\varphi\left(\tau^0\right)}{\partial\tau}d\tau,$$

then

$$\Xi\left(\tau^0+d\tau;\tau^0\right)\simeq\Xi_a\left(\tau^0+d\tau;\tau^0\right)\cos\left(\frac{\partial\varphi\left(\tau^0\right)}{\partial\tau}d\tau\right)^2,$$

$$\Xi_a\left(\tau;\tau^0\right)=\left(\alpha^0\right)^2\left|\frac{\mathbf{a}\left(\tau\right)^H\mathbf{a}\left(\tau^0\right)}{\left\|\mathbf{a}\left(\tau\right)\right\|}\right|^2, \quad (20)$$

where $\Xi_a\left(\tau;\tau^0\right)$ in (20) is the ambiguity function of the baseband signal $a\left(t\right)$, which is essential to predict and characterize the behaviour of $\widehat{\tau}$ (13) with respect to the SNR.

$$CRB_{\tau|\epsilon}^{-1}\left(\epsilon\right)=SNR_{out}F_s^2\left(\left(2\pi f_c/F_s\right)^2+\mathbf{a}^H\mathbf{V}\mathbf{a}/\mathbf{a}^H\mathbf{a}\right) \quad (21)$$

where $SNR_{out}=\frac{\left(\alpha^0\right)^2\left\|\mathbf{a}\left(\tau^0\right)\right\|^2}{\left(\sigma_n^0\right)^2/2}=\frac{2\left\|\mathbf{a}\right\|^2}{\left(\sigma_n^0\right)^2}\left(\alpha^0\right)^2$ is the SNR at the ouput of the MLE (13) (a.k.a matched filter [30], [34]) for the true value τ^0 of τ, and $\cos\left(\left(\partial\varphi\left(\tau^0\right)/\partial\tau\right)d\tau\right)^2=\cos\left(2\pi f_c d\tau\right)^2$. Then, if $f_c\gg B$, $\cos\left(2\pi f_c\left(\tau-\tau^0\right)\right)^2$ behaves like a Dirac comb which samples the baseband signal ambiguity function $\Xi_a\left(\tau;\tau^0\right)$ (20). Firstly, this sampling effect generates an ambiguity function $\Xi\left(\tau;\tau^0\right)$ with multiple local maxima within the main lobe of $\Xi_a\left(\tau;\tau^0\right)$ (see Fig. 1). Hence, since at high SNR, the MLE score function (16) tends to a sampled version of $\Xi_a\left(\tau;\tau^0\right)$, the MLE $\widehat{\tau}$ remains close to the maximum maximorum of $\Xi\left(\tau;\tau^0\right)$, that is close to τ^0, with a variance linked to the curbature of $\Xi\left(\tau;\tau^0\right)$ at the vicinity of τ^0 (18a), that is $CRB_{\tau|\epsilon}\left(\epsilon^0\right)$ (18b). This is highlighted in Fig. 2 displaying the MSE of MLE $\widehat{\tau}$ (13) and the $CRB_{\tau|\epsilon}\left(\epsilon\right)$ (21). Since $f_c/F_s\geq 1540/20=77\gg 1$, then (21) becomes simply $CRB_{\tau|\epsilon}\left(\epsilon\right)\simeq SNR_{out}\left(2\pi f_c\right)^2$ and depends only on the energy carried by the baseband signal and no longer on its temporal or frequential shape. Secondly, when the SNR decreases, at one point, the contribution of the noise in the MLE score function (16) allows to obtain the maximum in the closest local maximum, generating a "jump" of $1/f_c$ in the estimation error (see Fig. 1), and therefore a threshold effect on the MSE. As the SNR keeps on decreasing, the maximum of the MLE score function (16) will be located, due to the noise contribution, at other local maxima of smaller values, amounting to a random sampling of the baseband signal ambiguity function $\Xi_a\left(\tau;\tau^0\right)$. This behavior of the estimation error is similar to that of the estimation

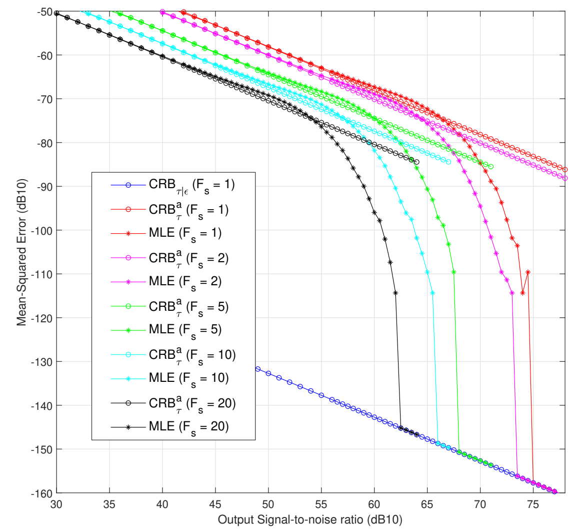

Fig. 2. MLE, $CRB_{\tau|\epsilon}$ (21) and CRB_{τ}^{a} (22) computed for $F_s\in\{1,2,5,10,20\}$. $CRB_{\tau|\epsilon}$ is not affected by changes in F_s.

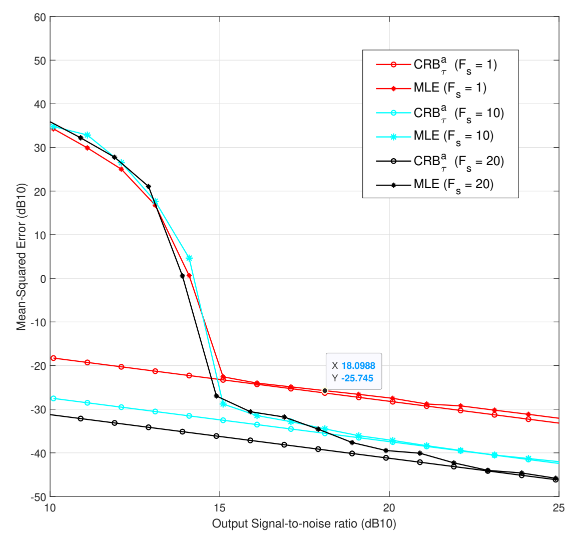

Fig. 3. MLE, CRB_{τ}^{a} (22) computed for for $F_s\in\{1,10,20\}$.

error of MLE $\widehat{\tau}_a$ of τ for the standard CSM (2) where the phase of the complex amplitude is regarded as free parameter: $\mathbf{x}=\alpha e^{j\phi}\mathbf{a}\left(\tau\right)+\mathbf{n}$ [24]. In this situation the MLE score function is $\left\|\Pi_{\mathbf{a}\left(\tau\right)}\mathbf{x}\right\|^2$ which tends to $\Xi_a\left(\tau;\tau^0\right)$ (20) at high SNR. Hence, in this region of operation, the MSE of $\widehat{\tau}$ is expected to converge towards the CRB of $\widehat{\tau}_a$

$$CRB_{\tau}^{a}=1/F^a,\ F^a=SNR_{out}F_s^2\left(\mathbf{a}^H\mathbf{V}\mathbf{a}/\mathbf{a}^H\mathbf{a}\right). \quad (22)$$

This is highlighted in Fig. 2 which also displays CRB_{τ}^{a} (22). Note that the curvature of the ambiguity function $\Xi_a\left(\tau;\tau^0\right)$ (20) of the code plays an important role in the SNR threshold value. For a given BPSK like signal $a(t)$, the curvature of $\Xi_a(\tau; \tau^0)$, which is inversely proportional to CRB_τ^a (22), depends on F_s, as it is illustrated in Fig. 1 ($f_c = 100/F_c$ to improve the readibility of plots).

Thirdly, as the SNR keeps on decreasing, a second well-known SNR threshold occurs leading to the last region of operation, the a priori region [28] as illustrated by Fig. 3.

V. Conclusion

The specific CSM considered here is representative of a class of nonlinear problems in which the likelihood function displays numerous ambiguities that are closely spaced. This observation suggests the presence of not just the well-known three regions of operation but rather five distinct regions within this class. From a practical perspective, this communication equips us with the necessary tools to evaluate the advantages of compensating for transmission phase when conducting delay estimation.

References

[1] D. A. Swick, "A Review of Wideband Ambiguity Functions," Naval Res. Lab., Washington DC, Tech. Rep. 6994, 1969.

[2] H. L. Van Trees, *Detection, Estimation, and Modulation Theory, Part III: Radar – Sonar Signal Processing and Gaussian Signals in Noise*. J. Wiley & Sons, 2001.

[3] U. Mengali and A. N. D'Andrea, *Synchronization Techniques for Digital Receivers*. New York, USA: Plenum Press, 1997.

[4] H. L. Van Trees, *Optimum Array Processing*. Wiley-Interscience, New-York, 2002.

[5] D. W. Ricker, *Echo Signal Processing*. Springer, New York, USA: Kluwer Academic, 2003.

[6] J. Chen, Y. Huang, and J. Benesty, "Time delay estimation," in *Audio Signal Processing for Next-Generation Multimedia Communication Systems*, Y. Huang and J. Benesty, Eds. Boston, MA, USA: Springer, 2004, ch. 8, pp. 197–227.

[7] B. C. Levy, *Principles of Signal Detection and Parameter Estimation*. Springer, 2008.

[8] F. B. D. Munoz, C. Vargas, and R. Enriquez, *Position Location Techniques and Applications*. Oxford, GB: Academic Press, 2009.

[9] J. Yan *et al.*, "Review of range-based positioning algorithms," *IEEE Trans. on AES*, vol. 28, no. 8, pp. 2–27, Aug. 2013.

[10] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*. Englewood Cliffs, New Jersey, USA: Prentice-Hall, 1993.

[11] H. L. V. Trees and K. L. Bell, Eds., *Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/Tracking*. NY, USA: Wiley/IEEE Press, 2007.

[12] P. Stoica and A. Nehorai, "Performances study of conditional and unconditional direction of arrival estimation," *IEEE Trans. on ASSP*, vol. 38, no. 10, pp. 1783–1795, Oct. 1990.

[13] A. Renaux, P. Forster, E. Chaumette, and P. Larzabal, "On the high-SNR conditional maximum-likelihood estimator full statistical characterization," *IEEE Trans. on SP*, vol. 54, no. 12, pp. 4840 – 4843, Dec. 2006.

[14] Q. Jin, K. M. Wong, and Z.-Q. Luo, "The Estimation of Time Delay and Doppler Stretch of Wideband Signals," *IEEE Trans. on SP*, vol. 43, no. 4, p. 904–916, April 1995.

[15] A. Dogandzic and A. Nehorai, "Cramér-Rao bounds for estimating range, velocity, and direction with an active array," *IEEE Trans. on SP*, vol. 49, no. 6, pp. 1122–1137, June 2001.

[16] N. Noels, H. Wymeersch, H. Steendam, and M. Moeneclaey, "True Cramér-Rao bound for timing recovery from a bandlimited linearly modulated waveform with unknown carrier phase and frequency," *IEEE Trans. on Com*, vol. 52, no. 3, pp. 473–483, March 2004.

[17] Y. S. W. He-Wen and W. Qun, "Influence of random carrier phase on true cramér-rao lower bound for time delay estimation," in *Proc. of the IEEE ICASSP*, Honolulu, USA, April 2007.

[18] J. Johnson and M. Fowler, "Cramér-Rao lower bound on Doppler frequency of coherent pulse trains," in *Proc. of the IEEE ICASSP*, Las Vegas, USA, March 2008.

[19] P. Closas, C. Fernández-Prades, and J. A. Fernández-Rubio, "Cramér-Rao Bound Analysis of Positioning Approaches in GNSS Receivers," *IEEE Trans. on SP*, vol. 57, no. 10, pp. 3775–3786, Nov. 2009.

[20] T. Zhao and T. Huang, "Cramér-Rao lower bounds for the joint delay-doppler estimation of an extended target," *IEEE Trans. on SP*, vol. 64, no. 6, pp. 1562–1573, March 2016.

[21] Y. Chen and R. S. Blum, "On the Impact of Unknown Signals on Delay, Doppler, Amplitude, and Phase Parameter Estimation," *IEEE Trans. on SP*, vol. 67, no. 2, pp. 431–443, Jan. 2019.

[22] D. Medina, J. Vilà-Valls, E. Chaumette, F. Vincent, and P. Closas, "Cramér-Rao bound for a mixture of real- and integer-valued parameter vectors and its application to the linear regression model," *Signal Processing*, vol. 179, 2021.

[23] P. C. C. X.X. Niu and Y. Chan, "Wavelet based approach for joint time delay and Doppler stretch measurements," *IEEE Trans. on AES*, vol. 35, no. 3, pp. 1111–1119, July 1999.

[24] P. Das, J. Vilà-Valls, F. Vincent, L. Davain, and E. Chaumette, "A New Compact Delay, Doppler Stretch and Phase Estimation CRB with a Band-Limited Signal for GenE. Remote Sensing Applications," *Remote Sensing*, vol. 12, no. 18, Sep. 2020.

[25] C. Lubeigt, L. Ortega, J. Vilà-Valls, L. Lestarquit, and E. Chaumette, "Joint delay-doppler estimation performance in a dual source context," *Remote Sensing*, vol. 12, no. 23, 2020.

[26] H. McPhee, L. Ortega, J. Vilà-Valls, and E. Chaumette, "Accounting for acceleration—signal parameters estimation performance limits in high dynamics applications," *IEEE Trans. on AES*, vol. 59, no. 1, pp. 610–622, 2023.

[27] A. Bartov and H. Messer, "Lower bound on the achievable DSP performance for localizing step-like continuous signals in noise," *IEEE Trans. on SP*, vol. 46, no. 8, pp. 2195–2201, Aug. 1998.

[28] R. D. J. V. Nee, J. Siereveld, P. C. Fenton, and B. R. Townsend, "Synchronization over rapidly time-varying multi-path channels for cdma downlink receiver in time-division mode," *IEEE Trans. on VT*, vol. 56, no. 4, pp. 2216–2225, July 2007.

[29] D. Medina, L. Ortega, J. Vilà-Valls, P. Closas, F. Vincent, and E. Chaumette, "Compact CRB for delay, doppler and phase estimation - application to GNSS SPP & RTK performance characterization," *IET RSN*, vol. 14, no. 10, pp. 1537–1549, Sep. 2020.

[30] M. I. Skolnik, *Radar handbook*. McGraw-Hill Education, 2008.

[31] B. Ottersten, M. Viberg, P. Stoica, and A. Nehorai, "Exact and Large Sample Maximum Likelihood Techniques for Parameter Estimation and Detection in Array Processing," in *Radar Array Processing*, S. Haykin, J. Litva, and T. J. Shepherd, Eds. Heidelberg: Springer-Verlag, 1993, ch. 4, pp. 99–151.

[32] T. Moore, B. Sadler, and R. Kozick, "Maximum-likelihood estimation, the cramér–rao bound, and the method of scoring with parameter constraints," *IEEE Trans. on SP*, vol. 56, no. 3, pp. 895–908, 2008.

[33] J. D. Gorman and A. O. Hero, "Lower bounds for parametric estimation with constraints," *IEEE Trans. on IT*, vol. 36, no. 6, pp. 1285–1301, 1990.

[34] P. J. G. Teunissen and O. Montenbruck, Eds., *Handbook of Global Navigation Satellite Systems*. Switzerland: Springer, 2017.