An Intrinsic Modified Cramér-Rao Bound on Lie Groups

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***Abstract*—The Modified Cramér-Rao Bound (MCRB) proves to be of significant importance in non-standard estimation scenarios, when in addition to unknown deterministic parameters to be estimated, observations also depend on random nuisance parameters. Given the interest of applications that involve estimation on Lie Groups (LGs), as well as the relevance of non-standard estimation problems in many practical scenarios, the main concern in this communication is to derive an intrinsic MCRB on LGs (LG-MCRB). For this purpose, a modified unbiasedness constraint must be defined, yielding a modified Barankin Bound. A closed-form formula of the LG-MCRB is then provided for a LG Gaussian model on $SO(2)$, representing $2D$ rotation matrices, while considering non-Gaussian random nuisance parameters. The validity of this expression is then assessed through numerical simulations, and compared with the intrinsic CRB on LGs for a simplified illustrative scenario, involving a concentrated Gaussian prior distribution on the random nuisance parameters.**

***Index Terms*—Intrinsic Modified Cramér-Rao Bound on Lie Groups, Non-standard Estimation on Lie Groups**

I. Introduction

Performance bounds are fundamental tools in any estimation problem. They are convenient when validating an estimator built from a statistical model, especially in situations where implementing the optimal estimator is not feasible. They also give insights on the ultimate achievable minimum mean squared error (MSE) that an estimator can reach for a given model. In the realm of deterministic parameters estimation, the Cramér-Rao bound (CRB) [1]–[4] is the most widely used and the easiest to compute, provided that a tractable form of the observations' likelihood is available and the considered estimator is unbiased. However, in many practical estimation problems [5]–[10], the likelihood does not solely depend on the sought deterministic parameters, but also on random nuisance parameters. Unfortunately, in such cases, the likelihood can only be computed by marginalizing with respect to these nuisance parameters. Thus, deriving a closed-form expression of the likelihood, and consequently the CRB, can become challenging, and inaccessible in a non-Gaussian context. To bypass this stumbling block, prior works [5], [6] introduced the *Modified CRB (MCRB)* on unknown deterministic parameters in the presence of random nuisance parameters.

The formulation of the standard CRB and MCRB inequalities is specific to observations and parameters lying on a Euclidean space. However, in the last decade, a number of applications in navigation, robotics and automatic control [11], [12] shed light on estimation problems where both the unknown parameters and observations are constrained to lie in a Riemannian manifold [13], [14], *e.g., matrix Lie Groups (LGs)*. For instance, in the context of autonomous navigation, the Lie groups characterizing rotation matrices, $SO(n)$ and rotation-translation matrices, $SE(n)$, are extensively employed to model the attitude and pose of dynamic systems. In this context, remote sensors such as RADAR or LIDAR provide measurements of rotation angles constrained to lie on $SO(2)$. Also, in the context of computer vision, the unknown homography between two images captured by a camera can be modelled using the Lie group of similarity $Sim(3)$, and estimated thanks to $SE(3)$ measurements provided by an odometer. To be consistent with the geometrical properties of these specific parameters and observations, it is crucial to derive intrinsic bounds taking into account their LG geometry.

In the context of matrix LGs, a seminal intrinsic bound was proposed in [15]–[18], but suffers from two main shortcomings: first, it only admits closed-form expressions for the LG $SO(n)$, and second the observations are restricted to lie in a Euclidean space. In [19], based on the formalism of the Barankin Bound and the *uniform unbiasedness condition* [20], an alternative intrinsic CRB that allows to overcome these shortcomings was derived. On the other hand, LG estimation problems with nuisance parameters is a crucial issue in various applications: i) in robotics [11], [21], estimating the robot's attitude or pose in LGs can be achieved with camera measurements that depend on the latent affine transformation between the observed 2D point and the 2D robot position; ii) in computer vision, the registration of medical images from different modalities or time-points implies an unknown similarity transformation between the observed images and the reference image, lying on $Sim(2)$ [22], and nuisance parameters encompass the measured 2D pixel points of the reference image; or iii) in radar tracking, sequentially estimating the attitude of an extended mobile target implies the existence of an unknown latent covariance matrix modelling the dispersion of radar measurements [23]. Then, to tackle this plethora of estimation problems, it is important to derive *intrinsic* bounds taking into account nuisance parameters in the LG modelling.

As a contribution to this framework, our concern is to derive a generalization on matrix LGs of the MCRB, so-called *Intrinsic MCRB* (LG-MCRB), for estimation problems with unknown deterministic LG parameters in the presence of LG random nuisance parameters. To achieve this, we leverage the demonstration of the LG-CRB developed in [19]. Since it is possible to perform integration on the LG product of the LG of observations G' and the LG of nuisance parameters G_r, we can define a joint distribution on the LG $G' \times G_r$. This property allows to deduce a *modified unbiasedness condition* and then an intrinsic modified Barankin bound (BB) on LGs (LG-MBB), with respect to modified observations lying on $G' \times G_r$. Then, with a similar reasoning as [19], we can determine the expression of the sought LG-MCRB. The explicit LG-MCRB formula is exemplified with a practical estimation problem for a LG Gaussian model on $SO(2)$ in the presence of non-Gaussian random nuisance parameters. In line with the discussions on the looseness of the MCRB compared to the CRB in the Euclidean framework [5]–[7], we compare the closed-form expressions of the proposed LG-MCRB bound with the LG-CRB by assuming, solely for illustration purposes, an uniform and a Gaussian prior on the random nuisance parameter.

II. Problem Statement

A. LG definitions

A matrix LG $G \subset \mathbb{R}^{n\times n}$ is a matrix space which respects the properties of smooth manifold and group. It implies the definition of a tangent space, also referred to as Lie algebra, and denoted $\mathfrak{g}$. The exponential and logarithmic maps, denoted respectively, $\mathrm{Exp}_G : \mathfrak{g} \to G$ and $\mathrm{Log}_G : G \to \mathfrak{g}$ associate each element of the LG to $\mathfrak{g}$. Since the latter is isomorphic to $\mathbb{R}^m$, we can define two bijections $[.]^\wedge : \mathbb{R}^m \to \mathfrak{g}$ and $[.]^\vee : \mathfrak{g} \to \mathbb{R}^m$. Thus, the exponential and logarithmic mappings can be reformulated as, $\forall\, \mathbf{a} \in \mathbb{R}^m$, $\mathrm{Exp}^\wedge_G(\mathbf{a}) = \mathrm{Exp}_G([\mathbf{a}]^\wedge_G)$ and $\forall\, \mathbf{X} \in G$, $[\mathrm{Log}_G(\mathbf{X})]^\vee_G = \mathrm{Log}^\vee_G(\mathbf{X})$. As an example, the Special Orthogonal group $SO(2)$ (later used in the illustrative example) is a Lie group of $2D$ rotation matrices $\mathbf{R}$, such that $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, where $^\top$ denotes the transpose operator. $SO(2)$ describes all possible rotations of a physical object in a plane. Let $\theta \in \mathbb{R}$ denote the rotation angle, the exponential and logarithmic applications are then defined as,

$$\mathrm{Exp}^\wedge_{SO(2)}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \mathbf{R}, \tag{1}$$

$$\mathrm{Log}^\vee_{SO(2)}(\mathbf{R}) = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}^\vee = \theta. \tag{2}$$

B. Integration and derivation on LGs

The properties of smooth manifold allow to easily define the concept of derivation by generalizing the notion of Euclidean directional derivative in an intrinsic way. Let $\mathbf{h} : G \to G'$ be a LG-valued function, the right Lie derivative of $\mathbf{h}$ on $\mathbf{X} \in G$ with dimension m, is defined as,

$$\mathcal{L}^R_{\mathbf{h}(\mathbf{X})} \overset{def}{=} \left.\frac{\partial \mathrm{Log}^\vee_G\left(\mathbf{h}(\mathbf{X})^{-1}\mathbf{h}(\mathbf{X}\,\mathrm{Exp}^\wedge_G(\boldsymbol{\delta}))\right)}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}. \tag{3}$$

wherein $\boldsymbol{\delta} \in \mathbb{R}^m$. Particularly, if $\mathbf{h}$ is real-valued, then $\mathrm{Log}^\vee_G(.) = \mathbf{I}$ (where $\mathbf{I}$ denotes the identity matrix with appropriate dimension). Furthermore, we can define the notion of integration. Indeed, the concept of volume forms can be defined for any manifold, and specifically for a LG it yields a group measure. Let $f : G \to \mathbb{R}^m$, the integration on LGs is defined as,

$$I = \int_G f(\mathbf{X})\, \lambda_G(d\mathbf{X}). \tag{4}$$

where λ_G denotes a left-invariant volume form, termed Haar measure [24]. This definition can straightforwardly be generalized to a multivariate function.

C. The Baker-Campbell-Hausdorff formula

The Baker-Campbell-Hausdorff (BCH) formula [25, Theorem 5.5] is a fundamental result in LGs providing an explicit expression for $\mathrm{Log}^\vee_G\left(\mathrm{Exp}^\wedge_G(\mathbf{a})\,\mathrm{Exp}^\wedge_G(\mathbf{b})\right) = \mathrm{BCH}(\mathbf{a},\mathbf{b}) \neq \mathbf{a}+\mathbf{b}$ since most LGs are non-commutative. A useful approximation of the BCH provides the following expression, $\forall\,(\mathbf{a},\mathbf{b}) \in \mathbb{R}^m \times \mathbb{R}^m$:

$$\mathrm{Log}^\vee_G\left(\mathrm{Exp}^\wedge_G(\mathbf{a})\,\mathrm{Exp}^\wedge_G(\mathbf{b})\right) = \mathbf{b} + \psi_G(\mathbf{b})\mathbf{a} + \mathcal{O}(\|\mathbf{a}\|^2). \tag{5}$$

where $\psi_G(\mathbf{b}) \overset{def}{=} \frac{ad_G(\mathbf{b})}{e^{ad_G(\mathbf{b})}-1}$ stands for the inverse of the left Jacobian matrix of G and $ad_G(.) : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is the adjoint representation on $\mathfrak{g}$ of the element $\mathbf{b}$.

D. Standard estimation on LGs

Let $\mathbf{Z} \in G' \subset \mathbb{R}^{n\times n}$ be a set of observations on a matrix LG G'. $\mathbf{Z}$ is connected to an unknown parameter $\mathbf{X}_0 \in G$ through the likelihood function $p(\mathbf{Z}|\mathbf{X}_0)$. For the sake of generality, let $\mathbf{H} : G \to G''$ be a LG-valued function, then we seek for an estimator of $\mathbf{H}(\mathbf{X}_0)$ from observations $\mathbf{Z}$, denoted $\widehat{\mathbf{H}(\mathbf{X}_0)}$. An essential question is, "how to assess the discrepancy between $\mathbf{H}(\mathbf{X}_0)$ and $\widehat{\mathbf{H}(\mathbf{X}_0)}$?". An intrinsic metric classically used in LG estimation problems [16], [19], [21] is,

$$\boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0), \widehat{\mathbf{H}(\mathbf{X}_0)}\right) \overset{def}{=} \mathrm{Log}^\vee_{G''}\left(\mathbf{H}(\mathbf{X}_0)^{-1}\widehat{\mathbf{H}(\mathbf{X}_0)}\right). \tag{6}$$

referred to as $\boldsymbol{l}^{(0)}_{G''} \overset{nota.}{=} \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0), \widehat{\mathbf{H}(\mathbf{X}_0)}\right)$ in the sequel for the sake of clarity. Two fundamental quantities can be derived from (6) [19],

- The intrinsic bias defined as the mean value of the intrinsic gap (6),

$$\mathbf{b}_{\mathbf{Z}|\mathbf{X}_0}\left(\mathbf{H}(\mathbf{X}_0), \widehat{\mathbf{H}(\mathbf{X}_0)}\right) \overset{def}{=} \int_{G'} \boldsymbol{l}^{(0)}_{G''} p(\mathbf{Z}|\mathbf{X}_0)\lambda_{G'}(d\mathbf{Z}), \\ \overset{nota.}{=} \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\boldsymbol{l}^{(0)}_{G''}\right). \tag{7}$$

- The intrinsic MSE (LG-MSE) defined as the correlation matrix of the intrinsic gap (6),

$$\mathbf{C}_{\mathbf{Z}|\mathbf{X}_0}\left(\mathbf{H}(\mathbf{X}_0), \widehat{\mathbf{H}(\mathbf{X}_0)}\right) \overset{def}{=} \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\boldsymbol{l}^{(0)}_{G''}\boldsymbol{l}^{(0)\top}_{G''}\right). \tag{8}$$

E. Non-standard estimation on LGs

Unlike standard estimation problems for which a closed-form expression of $p(\mathbf{Z}|\mathbf{X}_0)$ exists, we tackle in this work a non-standard estimation problem where the latter is only accessible through an integral form, i.e., $\mathbf{Z}$ is connected to an unknown parameter vector $\mathbf{X}_0 \in G$ through a "compound" probability density function (pdf) $p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)$, where $\mathbf{Y} \in G_r$ are random *nuisance parameters* on a matrix LG G_r. In general the pdf of $\mathbf{Y}$ might depend on $\mathbf{X}_0$, $p(\mathbf{Y}|\mathbf{X}_0)$, then

$$p(\mathbf{Z}|\mathbf{X}_0) = \int_{G_r} p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)\lambda_{G_r}(d\mathbf{Y}). \tag{9}$$

Unfortunately, this marginalization is in general mathematically intractable. This is why we focus on the joint pdf between observations and the random nuisance parameters, $p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)$. In this framework, the intrinsic bias (7) and covariance (8) metrics must be extended to the LG $G' \times G_r$.

III. Intrinsic Modified Cramér-Rao Lower Bound

In this section, we first recall in III-A the derivations of the intrinsic BB on LGs (LG-BB), established in [19], to make then a further extension of these results to address non-standard estimation problems on LGs in III-B.

A. Review on the intrinsic Barankin Bound and LG-CRB

Recall that, for standard estimation problems, $\mathbf{Z}$ is a set of observations on G' depending on the unknown parameter $\mathbf{X}_0$ on G with dimension m, and characterized by the likelihood function $p(\mathbf{Z}|\mathbf{X}_0)$. $\mathbf{H}$ is a smooth function on $G \to G''$. To define a lower bound, a fundamental property in the Euclidean case is the strict-sense/uniform unbiasedness. This property is well-known in the Euclidean framework and allows to theorize the BB [20]. An intrinsic formulation of this constraint on the LG estimator $\widehat{\mathbf{H}(\mathbf{X}_0)}$, for standard estimation, is [19],

$$\mathbf{b}_{\mathbf{Z}|\mathbf{X}}\left(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)}\right) = \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}(\mathbf{X})\right),\ \forall\, \mathbf{X} \in G. \tag{10}$$

In a similar fashion as the BB in the Euclidean space:

Definition 3.1 (LG-BB on LG): The LG-BB, denoted $\mathbf{P}_{\text{LG-BB}}$, is defined as the minimum value of the intrinsic MSE (8) under the intrinsic uniform unbiasedness constraint (10),

$$\begin{aligned}\mathbf{P}_{\text{LG-BB}} &= \min_{\widehat{\mathbf{H}(\mathbf{X}_0)}} \mathbf{C}_{\mathbf{Z}|\mathbf{X}_0}(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)})\\ &\text{s.t. } \mathbf{b}_{\mathbf{Z}|\mathbf{X}}\left(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)}\right) = \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}(\mathbf{X})\right),\\ &\forall\, \mathbf{X} \in G.\end{aligned} \tag{11}$$

A lower bound $\mathbf{P}$ on the LG-MSE is then derived from a discretization of the constraint (10) on a set of test points $\mathbf{X}^{(1:L)} = \{\mathbf{X}^{(1)},\dots,\mathbf{X}^{(L)}\} \in G$ yielding the inequality,

$$\mathbf{C}_{\mathbf{Z}|\mathbf{X}_0}(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)}) \succeq \mathbf{P},\ \mathbf{P} = \boldsymbol{\Delta}_G\, \mathbf{R}_{\mathbf{v}_{\mathbf{X}_0}}^{-1} \boldsymbol{\Delta}_G^\top. \tag{12}$$

where $\succeq$ means that $\mathbf{C}_{\mathbf{Z}|\mathbf{X}_0}(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)}) - \mathbf{P}$ is positive semi-definite (Löwner ordering [26]), and

$$\boldsymbol{\Delta}_G^\top \stackrel{def}{=} \begin{bmatrix} \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}(\mathbf{X}^{(1)})\right)^\top \\ \vdots \\ \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}(\mathbf{X}^{(L)})\right)^\top \end{bmatrix}, \tag{13}$$

$$\mathbf{R}_{\mathbf{v}_{\mathbf{X}_0}} = \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z}|\mathbf{X}^{(1:L)}\right)\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z}|\mathbf{X}^{(1:L)}\right)^\top\right).$$

with $\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z}|\mathbf{X}^{(1:L)}\right) = \left[v_{\mathbf{X}_0}\left(\mathbf{Z}|\mathbf{X}^{(1)}\right),\dots,v_{\mathbf{X}_0}\left(\mathbf{Z}|\mathbf{X}^{(L)}\right)\right]^\top$ is the vector gathering the likelihood ratios $v_{\mathbf{X}_0}\left(\mathbf{Z}|\mathbf{X}^{(l)}\right) = \frac{p(\mathbf{Z}|\mathbf{X}^{(l)})}{p(\mathbf{Z}|\mathbf{X}_0)}$, $\forall l \in \{1,\dots,L\}$.

Definition 3.2 (LG-CRB): The inequality (12) is the cornerstone for deriving the LG-CRB; selecting the test points

$$\mathbf{X}^{(1:L)} = \{\mathbf{X}_0, \mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\mathbf{i}_1\,\delta_1),..,\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\mathbf{i}_{L-1}\,\delta_{L-1})\},$$

$$\mathbf{i}_l = \Big[\, 0,\dots,\underbrace{1}_{l\text{th component}},\dots,0 \,\Big]^\top \in \mathbb{R}^m. \tag{14}$$

yields the definition of the LG-CRB, when δ_l tends to 0,

$$\begin{aligned}\mathbf{P}_{\text{LG-CRB}} &= \mathcal{L}^R_{\mathbf{H}(\mathbf{X}_0)}\mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\mathbf{s}(\mathbf{Z}|\mathbf{X}_0)\mathbf{s}(\mathbf{Z}|\mathbf{X}_0)^\top\right)^{-1}\\ &\quad\times(\mathcal{L}^R_{\mathbf{H}(\mathbf{X}_0)})^\top.\end{aligned} \tag{15}$$

where, $\mathbf{s}(\mathbf{Z}|\mathbf{X}_0) = \left.\frac{\partial \log p\left(\mathbf{Z}|\mathbf{X}_0\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})\right)}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}$, and $\mathcal{L}^R_{\mathbf{H}(\mathbf{X}_0)}$ denotes the right Lie derivative of $\mathbf{H}$ according to $\mathbf{X}_0$ (3).

B. Derivation of the intrinsic MCRB (LG-MCRB)

Theorem 1 (Intrinsic modified unbiasedness constraint for non-standard estimation): The constraint previously defined in (10), can be extended to non-standard estimation as: $\forall \mathbf{X} \in G$,

$$\mathbf{b}_{\mathbf{Z},\mathbf{Y}|\mathbf{X}_0}\left(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)}\right) = \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}(\mathbf{X})\right). \tag{16}$$

where the intrinsic bias is now defined w.r.t. to $p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)$.

Proof. 1) First, assuming that the support of $p(\mathbf{Z},\mathbf{Y}|\mathbf{X})$ is independent of $\mathbf{X}$, we have $\forall \mathbf{X} \in G$,

$$\begin{aligned}\mathbb{E}_{p(\mathbf{Z},\mathbf{Y}|\mathbf{X})}\left(\boldsymbol{l}^{(0)}_{G''}\right) &= \int_{G''} \boldsymbol{l}^{(0)}_{G''} p(\mathbf{Z},\mathbf{Y}|\mathbf{X})\lambda_G(d\mathbf{Z},d\mathbf{Y}),\\ &= \int_{G''} \boldsymbol{l}^{(0)}_{G''}\frac{p(\mathbf{Z},\mathbf{Y}|\mathbf{X})}{p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)} p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)\lambda_G(d\mathbf{Z},d\mathbf{Y}),\\ &= \mathbb{E}_{p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)}\left(\boldsymbol{l}^{(0)}_{G''} v_{\mathbf{X}_0}\left(\mathbf{Z},\mathbf{Y}|\mathbf{X}\right)\right).\end{aligned} \tag{17}$$

where the ratio is now considered w.r.t. the joint pdf $p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)$, i.e., $v_{\mathbf{X}_0}\left(\mathbf{Z},\mathbf{Y}|\mathbf{X}\right) = \frac{p(\mathbf{Z},\mathbf{Y}|\mathbf{X})}{p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)}$.

2) Second, for any estimator $\widehat{\mathbf{H}(\mathbf{X}_0)}$ verifying the constraint (10), we have $\forall \mathbf{X} \in G$:

$$\begin{aligned}\boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}(\mathbf{X})\right) &= \mathbf{b}_{\mathbf{Z}|\mathbf{X}}\left(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)}\right),\\ &\stackrel{def}{=} \mathbb{E}_{p(\mathbf{Z}|\mathbf{X})}\left(\boldsymbol{l}^{(0)}_{G''}\right),\\ &= \int_{G''} \boldsymbol{l}^{(0)}_{G''} p(\mathbf{Z}|\mathbf{X})\lambda_{G''}(d\mathbf{Z}).\end{aligned} \tag{18}$$

Moreover, by applying the marginalization formula to (18),

$$\begin{aligned} p(\mathbf{Z}|\mathbf{X}) &= \int_{G_r} p(\mathbf{Z},\mathbf{Y}|\mathbf{X})\lambda_{G_r}(d\mathbf{Y}), \\ (18) &= \int_{G''} l^{(0)}_{G''} p(\mathbf{Z}|\mathbf{X})\lambda_{G''}(d\mathbf{Z}), \\ &= \int_{G''\times G_r} l^{(0)}_{G''} p(\mathbf{Z},\mathbf{Y}|\mathbf{X})\lambda_{G''\times G_r}(d\mathbf{Z},d\mathbf{Y}), \\ &\overset{def}{=} \mathbb{E}_{p(\mathbf{Z},\mathbf{Y}|\mathbf{X})}\left(\boldsymbol{l}^{(0)}_{G''}\right). \end{aligned} \quad (19)$$

As detailed in (17), introducing the ratio $v_{\mathbf{X}_0}(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)$, $\mathbb{E}_{p(\mathbf{Z},\mathbf{Y}|\mathbf{X})}\left(\boldsymbol{l}^{(0)}_{G''}\right)$ can be recast as,

$$\begin{aligned} \mathbb{E}_{p(\mathbf{Z},\mathbf{Y}|\mathbf{X})}\left(\boldsymbol{l}^{(0)}_{G''}\right) &= \mathbb{E}_{p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)}\left(\boldsymbol{l}^{(0)}_{G''} v_{\mathbf{X}_0}\left(\mathbf{Z},\mathbf{Y}|\mathbf{X}\right)\right), \\ &\overset{nota.}{=} \mathbf{b}_{\mathbf{Z},\mathbf{Y}|\mathbf{X}_0}\left(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)}\right). \end{aligned}$$

Finally the modified unbiasedness constraint follows from combining 1) and 2), where the intrinsic bias is now defined w.r.t. to $p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)$,

$$\begin{aligned} &\mathbf{b}_{\mathbf{Z},\mathbf{Y}|\mathbf{X}_0}\left(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)}\right) \\ &\overset{def}{=} \mathbb{E}_{p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)}\left(\boldsymbol{l}^{(0)}_{G''} v_{\mathbf{X}_0}\left(\mathbf{Z},\mathbf{Y}|\mathbf{X}\right)\right), \forall \mathbf{X}\in G. \end{aligned} \quad (20)$$

□

Consequently, the minimization problem subject to the constraint (16) can be extended by considering the joint pdf $p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)$ when defining the LG-MSE,

$$\mathbf{C}_{\mathbf{Z},\mathbf{Y}|\mathbf{X}_0}(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)}) \overset{def}{=} \mathbb{E}_{p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)}\left(\boldsymbol{l}^{(0)}_{G''}\left(\boldsymbol{l}^{(0)}_{G''}\right)^{\top}\right). \quad (21)$$

Theorem 2 (Intrinsic Modified BB on LGs (LG-MBB)): The LG-MBB, denoted $\mathbf{P}_{\text{LG-MBB}}$, can be defined as the minimum value of the modified intrinsic MSE under the modified uniform ubiasedness constraint (16) in Theorem 1.

$$\begin{aligned} &\mathbf{P}_{\text{LG-MBB}} \overset{def}{=} \min_{\widehat{\mathbf{H}(\mathbf{X}_0)}} \mathbf{C}_{\mathbf{Z},\mathbf{Y}|\mathbf{X}_0}(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)}) \quad (22) \\ &\text{s.t. } \mathbf{b}_{\mathbf{Z},\mathbf{Y}|\mathbf{X}_0}\left(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)}\right) = \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}(\mathbf{X})\right), \\ &\forall\ \mathbf{X}\in G. \end{aligned}$$

Then, following a similar reasoning as the LG-CRB fully derived in [19] and sketched in III-A, we define a LG-MCRB:

Definition 3.3 (Intrinsic MCRB on LG (LG-MCRB)): The LG-MCRB that approximates $\mathbf{P}_{\text{LG-MBB}}$ for a discrete set of test points of the form (14) is defined as,

$$\begin{aligned} \mathbf{P}_{\text{LG-MCRB}} = \mathcal{L}^R_{\mathbf{H}(\mathbf{X}_0)}\mathbb{E}_{p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)}\left(\mathbf{s}(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)\mathbf{s}(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)^{\top}\right)^{-1} \\ \times(\mathcal{L}^R_{\mathbf{H}(\mathbf{X}_0)})^{\top}. \end{aligned} \quad (23)$$

where, $\mathbf{s}(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0) = \left.\frac{\partial \log p\left(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0\text{Exp}^{\wedge}_G(\boldsymbol{\delta})\right)}{\partial\boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}$, and $\mathcal{L}^R_{\mathbf{H}(\mathbf{X}_0)}$ is previously defined in (3).

Proof. Following similar lines as [19], a lower bound on this constrained minimization problem for a set of points $\mathbf{X}^{(1:L)}$ verifying the constraint (16), yields the following inequality,

$$\mathbf{C}_{\mathbf{Z},\mathbf{Y}|\mathbf{X}_0}(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)}) \succeq \boldsymbol{\Delta}_G\left(\mathbf{R}'_{\mathbf{v}_{\mathbf{X}_0}}\right)^{-1}\boldsymbol{\Delta}_G^{\top}. \quad (24)$$

where applying (16) on $\mathbf{X}^{(1:L)}$ yields the matrix of constraints $\boldsymbol{\Delta}_G$, with $\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z},\mathbf{Y}|\mathbf{X}^{(1:L)}\right) = \left[v_{\mathbf{X}_0}\left(\mathbf{Z},\mathbf{Y}|\mathbf{X}^{(1)}\right),\dots,v_{\mathbf{X}_0}\left(\mathbf{Z},\mathbf{Y}|\mathbf{X}^{(L)}\right)\right]^{\top}$,

$$\begin{aligned} \boldsymbol{\Delta}_G^{\top} &\overset{def}{=} \begin{bmatrix} \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}(\mathbf{X}^{(1)})\right)^{\top} \\ \vdots \\ \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}(\mathbf{X}^{(L)})\right)^{\top} \end{bmatrix}, \\ &= \mathbb{E}_{p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)}\left(\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z},\mathbf{Y}|\mathbf{X}^{(1:L)}\right)\boldsymbol{l}^{(0)\top}_{G''}\right). \\ \mathbf{R}'_{\mathbf{v}_{\mathbf{X}_0}} &= \\ &\mathbb{E}_{p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)}\left(\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z},\mathbf{Y}|\mathbf{X}^{(1:L)}\right)\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z},\mathbf{Y}|\mathbf{X}^{(1:L)}\right)^{\top}\right). \end{aligned}$$

Following similar lines of proof of Theorem 3.3.1 in [19, §3.3], using the inequality (24) on test points of the form (14), yields the definition of the sought LG-MCRB. □

Important particular case: In the particular case of unimodular LGs [27] (such as $SO(n)$ and $SE(n)$), provided that the function $\boldsymbol{\delta}\to\log p\left(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0\text{Exp}^{\wedge}_G(\boldsymbol{\delta})\right)$ is sufficiently regular, the aforementioned expression can be further simplified,

$$\begin{aligned} &\mathbb{E}_{p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)}\left(\mathbf{s}(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)\,\mathbf{s}(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)^{\top}\right) = -\mathbb{E}_{p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0)} \\ &\left(\left.\frac{\partial^2\log p(\mathbf{Z},\mathbf{Y}|\mathbf{X}_0\,\text{Exp}^{\wedge}_G(\boldsymbol{\delta}_1)\,\text{Exp}^{\wedge}_G(\boldsymbol{\delta}_2))}{\partial\,\boldsymbol{\delta}_1\,\partial\,\boldsymbol{\delta}_2}\right|_{\boldsymbol{\delta}_1,\boldsymbol{\delta}_2=\mathbf{0}}\right). \end{aligned} \quad (25)$$

This formula is useful as it allows to derive closed-form expressions of the LG-MCRB, by injecting (25) in (23).

IV. Closed-Form Expressions

A. Considered problem

For the sake of illustration, we consider an odometer system embedded with a camera, measuring its rotation $\mathbf{X}_c\in SO(2)$ w.r.t. a fixed world frame. The camera captures N measurements of the position of a landmark represented with an orientation $\{\mathbf{Y}^{(i)}_p\}_{i=1}^N\in SO(2)$. The aim is to infer the unknown relative orientation $\mathbf{X}_{cp}$ between camera and the landmark. We use the information provided by $\mathbf{X}^{(i)}_c$, while taking into account the latent rotation $\mathbf{Y}^{(i)}_p$.

In other words, $\mathbf{X}^{(i)}_c$ can be decomposed as $\mathbf{X}^{(i)}_c = \mathbf{X}_{cp}\mathbf{Y}^{(i)}_p$. The uncertainty on the measurement of $\mathbf{X}^{(i)}_c$ can be modeled by a Gaussian distribution on $SO(2)$, yielding the following observation model,

$$\mathbf{Z}^{(i)} = \mathbf{X}_{cp}\mathbf{Y}^{(i)}_p\text{Exp}^{\wedge}_{SO(2)}\left(n_i\right), \quad n_i\sim\mathcal{N}(0,\sigma_n^2). \quad (26)$$

As a matter of fact, the latent variable $\mathbf{Y}^{(i)}_p$ is classically estimated by a tracking algorithm taking advantage of the camera pixel measurements [28]. In this dynamic context, note that a recent study [29] has proposed a recursive posterior

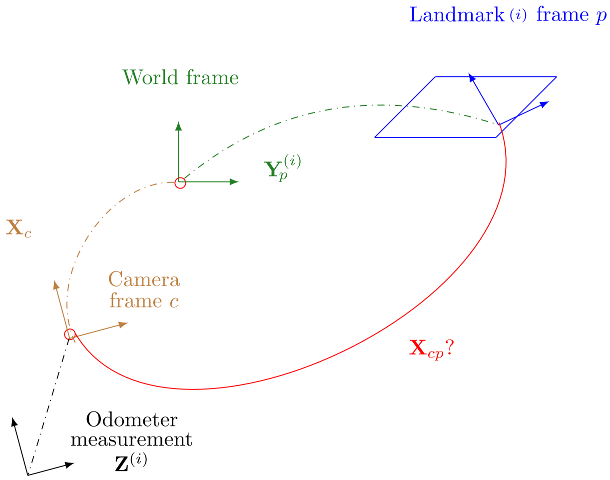


Fig. 1: Schematic of the illustrative example

Cramér-Rao bound on matrix Lie Groups. It follows that $\mathbf{Y}_p^{(i)}$ is random with LG-mean $\mathbf{I}$ and with variance depending on the algorithms' estimation error.
In the following, we are interested in deriving a Maximum Likelihood (ML) estimator of $\mathbf{X}_{cp}$, and then computing its LG-MCRB in the presence of $\mathbf{Y}_p^{(i)}$ depending on its uncertainty modeled by a prior distribution.

B. Maximum likelihood estimator

The standard LG-ML is obtained by maximizing the likelihood of the independent observations $\mathbf{Z} = \{\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(I)}\}$ depending on $\mathbf{Y}_p = \{\mathbf{Y}_p^{(1)}, \ldots, \mathbf{Y}_p^{(I)}\}$ also independent of each other. Nevertheless, as the latter are also unknown, it yields a non-standard LG-ML estimator of $\mathbf{X}_{cp}$ [7, §IV],

$$\left(\widehat{\mathbf{X}}_{cp}, \widehat{\mathbf{Y}}_p\right) = \underset{\mathbf{X}_{cp}, \mathbf{Y}_p}{\arg\max}\, p(\mathbf{Z}|\mathbf{Y}_p, \mathbf{X}_{cp}), \tag{27}$$

$$= \underset{\mathbf{X}_{cp}, \mathbf{Y}_p}{\arg\max} \prod_{i=1}^{N} p(\mathbf{Z}^{(i)}|\mathbf{Y}_p^{(i)}, \mathbf{X}_{cp}). \tag{28}$$

Due to the presence of $\mathbf{Y}_p^{(i)}$, (26) is under-determined and it is not possible to deduce an analytical solution of $\widehat{\mathbf{X}}_{cp}$. To overcome that, we take advantage of the commutativity of $SO(2)$ and apply the operator $\text{Log}^{\vee}_{SO(2)}(\cdot)$ on (26). It yields:

$$\text{Log}^{\vee}_{SO(2)}\left(\mathbf{Z}^{(i)}\right) = \text{Log}^{\vee}_{SO(2)}(\mathbf{X}_{cp}) + \text{Log}^{\vee}_{SO(2)}\left(\mathbf{Y}_p^{(i)}\right) + n_i. \tag{29}$$

Then, by noting $z^{(i)} = \text{Log}^{\vee}_{SO(2)}(\mathbf{Z}^{(i)})$, $y^{(i)} = \text{Log}^{\vee}_{SO(2)}\left(\mathbf{Y}_p^{(i)}\right)$ and $x_{cp} = \text{Log}^{\vee}_{SO(2)}(\mathbf{X}_{cp})$, we propose to find $\widehat{\mathbf{X}}_{cp}$ verifying the new following problem:

$$\widehat{x}_{cp} = \underset{x_{cp}}{\text{argmax}} \prod_{i=1}^{N} p(z^{(i)}|y^{(i)}, x_{cp}), \tag{30}$$

$$= \underset{x_{cp}}{\text{argmax}} \prod_{i=1}^{N} \mathcal{N}(x_{cp} + \mathbb{E}\left(y^{(i)}\right), \sigma_n^2). \tag{31}$$

$$\widehat{\mathbf{X}}_{cp} = \text{Exp}^{\wedge}_{SO(2)}(\widehat{x}_{cp}). \tag{32}$$

As the intrinsic mean of $\mathbf{Y}^{(i)}$ is $\mathbf{I}$, $\mathbb{E}\left(\text{Log}^{\vee}_{SO(2)}\left(\mathbf{Y}_p^{(i)}\right)\right) = 0$, it is straightforward to deduce:

$$\widehat{\mathbf{X}}_{cp} = \text{Exp}^{\wedge}_{SO(2)}\left(\frac{1}{N}\sum_{i=1}^{N} z^{(i)}\right). \tag{33}$$

Remark: Even though we make use of the commutativity of $SO(2)$ for simplicity, the reasoning behind deriving a LG-ML of $\mathbf{X}_{cp}$ can translate to non-commutative groups such as $SO(3)$ by generalizing the model (26). Indeed, to compute $\text{Log}^{\vee}_{SO(3)}(\mathbf{Z}^{(i)})$, we now need to apply the BCH formula (5) which makes appear $\psi_{SO(3)}\left(\text{Log}^{\vee}_{SO(3)}(\mathbf{n}_i)\right)$ where $\mathbf{n}_i$ is the observation noise. If we assume the variance noise is low, the latter can be approximated by the identity matrix yielding the same estimator (33) provided that the intrinsic mean of $\mathbf{Y}_p^{(i)}$ is $\mathbf{I}$.

C. LG-MCRB for a uniform random nuisance parameter

In the following, we consider that the knowledge of the parameter $\mathbf{Y}_p^{(i)}$ is provided with some level of uncertainty, modeled by a random uniform noise projected on $SO(2)$ defined for any angle $\theta \in [-\pi, \pi]$,

$$\mathbf{Y}_p^{(i)} = \mathbf{I} \times \text{Exp}^{\wedge}_{SO(2)}(b_i), \quad b_i \sim \mathcal{U}([-\theta, \theta]). \tag{34}$$

Due to the presence of uniform noise in the expression of $\mathbf{Y}_p$, the likelihood $p(\mathbf{Z}|\mathbf{X}_{cp})$ can not be analytically derived. Thus, we tackle a non-standard estimation problem for $\mathbf{X}_{cp}$. In order to derive a lower bound on the estimation of $\mathbf{X}_{cp}$ from $\mathbf{Z}$, in the presence of the random nuisance parameter $\mathbf{Y}_p$, we will make use of the LG-MCRB defined in (23). More specifically, we will use the expression in (25) since $SO(2)$ is unimodular. Let, $\delta_1, \delta_2 \in \mathbb{R}$, and denote the intrinsic modified Fisher information as J_1, then:

$$J_1 = \mathbb{E}_{p(\mathbf{Z}, \mathbf{Y}_p|\mathbf{X}_{cp})}(a),$$
$$a = \left.\frac{\partial^2 \log p(\mathbf{Z}, \mathbf{Y}_p|\mathbf{X}_{cp}\, \text{Exp}^{\wedge}_{SO(2)}(\delta_1)\, \text{Exp}^{\wedge}_{SO(2)}(\delta_2))}{\partial\, \delta_1\, \partial\, \delta_2}\right|_{\delta_1, \delta_2 = 0}.$$

In order to compute J_1, we need to define $p(\mathbf{Z}, \mathbf{Y}_p|\mathbf{X}_{cp})$,

$$p(\mathbf{Z}, \mathbf{Y}_p|\mathbf{X}_{cp}) = p(\mathbf{Z}|\mathbf{Y}_p, \mathbf{X}_{cp})p(\mathbf{Y}_p),$$

It is important to remark here that we do not need to know $p(\mathbf{Y}_p)$ to compute the LG-MCRB, but only $p(\mathbf{Z}|\mathbf{Y}_p, \mathbf{X}_{cp})$,

$$p(\mathbf{Z}|\mathbf{Y}_p, \mathbf{X}_{cp}) \propto \exp\left(-\frac{1}{2}\sum_{i=1}^{N} \frac{l_{SO(2)}(\mathbf{X}_{cp}\mathbf{Y}_p^{(i)}, \mathbf{Z}^{(i)})^2}{\sigma_n^2}\right), \tag{35}$$

and therefore,

$$a = \left.\frac{\partial^2 \log p(\mathbf{Z}|\mathbf{Y}_p, \mathbf{X}_{cp}\, \text{Exp}^{\wedge}_{SO(2)}(\delta_1)\, \text{Exp}^{\wedge}_{SO(2)}(\delta_2))}{\partial\, \delta_1\, \partial\, \delta_2}\right|_{\delta_1, \delta_2 = 0}.$$

Furthermore,

$$\log p(\mathbf{Z}|\mathbf{Y}_p, \mathbf{X}_{cp}\, \mathrm{Exp}^{\wedge}_{SO(2)}(\delta_1)\, \mathrm{Exp}^{\wedge}_{SO(2)}(\delta_2)) = -\frac{1}{2}\sum_{i=1}^{N} \frac{l_{SO(2)}(\mathbf{X}_{cp}\mathrm{Exp}^{\wedge}_{SO(2)}(\delta_1)\, \mathrm{Exp}^{\wedge}_{SO(2)}(\delta_2)\, \mathbf{Y}_p^{(i)}, \mathbf{Z}^{(i)})^2}{\sigma_n^2}, = -\frac{1}{2}\sum_{i=1}^{N} \frac{l_{SO(2)}(\mathbf{X}_{cp}\mathbf{Y}_p^{(i)}\mathrm{Exp}^{\wedge}_{SO(2)}(\delta_1+\delta_2)\,, \mathbf{Z}^{(i)})^2}{\sigma_n^2}. \tag{36}$$

Note that the expression (36) stems from $SO(2)$ being commutative: we have $\mathrm{Exp}^{\wedge}_{SO(2)}(\delta_1)\,\mathrm{Exp}^{\wedge}_{SO(2)}(\delta_2) = \mathrm{Exp}^{\wedge}_{SO(2)}(\delta_1+\delta_2)$. In addition,

$$\mathbf{X}_{cp}\mathrm{Exp}^{\wedge}_{SO(2)}(\delta_1+\delta_2)\, \mathbf{Y}_p^{(i)} = \mathbf{X}_{cp}\mathbf{Y}_p^{(i)}\mathrm{Exp}^{\wedge}_{SO(2)}(\delta_1+\delta_2).$$

The derivative of (36) can be obtained in two steps: 1) First, consider the first-order Taylor expansion of $l_{SO(2)}\left(\mathbf{X}_{cp}\mathbf{Y}_p^{(i)}\mathrm{Exp}^{\wedge}_{G}(\delta), \mathbf{Z}^{(i)}\right)$, given by

$$l_{SO(2)}\left(\mathbf{X}_{cp}\mathbf{Y}_p^{(i)}, \mathbf{Z}^{(i)}\right) + \left.\frac{\partial l_{SO(2)}\left(\mathbf{X}_{cp}\mathbf{Y}_p^{(i)}\mathrm{Exp}^{\wedge}_{SO(2)}(\epsilon), \mathbf{Z}^{(i)}\right)}{\partial \epsilon}\right|_{\epsilon=0} \delta + \mathcal{O}(\delta^2).$$

2) Using the BCH formula (5),

$$\left.\frac{\partial l_{SO(2)}\left(\mathbf{X}_{cp}\mathbf{Y}_p^{(i)}\mathrm{Exp}^{\wedge}_{SO(2)}(\epsilon), \mathbf{Z}^{(i)}\right)}{\partial \epsilon}\right|_{\epsilon=0} = -\psi_{SO(2)}\left(l_{SO(2)}\left(\mathbf{X}_{cp}\mathbf{Y}_p^{(i)}, \mathbf{Z}^{(i)}\right)\right), \tag{37}$$

Since $SO(2)$ is commutative, we have $\psi_{SO(2)}(.) = 1$. Thus, combining steps 1) and 2), (36) becomes:

$$\log p(\mathbf{Z}|\mathbf{Y}_p, \mathbf{X}_{cp}\, \mathrm{Exp}^{\wedge}_{SO(2)}(\delta_1)\, \mathrm{Exp}^{\wedge}_{SO(2)}(\delta_2)) = -\frac{1}{2}\sum_{i=1}^{N} \frac{\left(l_{SO(2)}(\mathbf{X}_{cp}\mathbf{Y}_p^{(i)}, \mathbf{Z}^{(i)}) - (\delta_1+\delta_2)\right)^2}{\sigma_n^2} + \mathcal{O}\left((\delta_1+\delta_2)^2\right) \tag{38}$$

By double differentiating (38) w.r.t. to δ_1, δ_2, and by taking $\delta_1 = \delta_2 = 0$, we obtain, $a = \sum_{i=1}^{N} \frac{1}{\sigma_n^2}$, therefore $J_1 = \frac{N}{\sigma_n^2}$ and the LG-MCRB for this model is,

$$P_{\text{LG-MCRB}} \overset{def}{=} J_1^{-1} = \frac{\sigma_n^2}{N} \tag{39}$$

Remark: Even though we exemplify the utilization of the bound for $SO(2)$, the steps to derive the LG-MCRB for other non-commutative Lie groups such as $SE(2)$ or $SO(3)$ are still valid. In general for a set of independent observations such that $\mathbf{Z}^{(i)} = \mathbf{X}_{cp}\mathbf{Y}_p^{(i)}\mathrm{Exp}^{\wedge}_{G}(\mathbf{n}_i)$, $\mathbf{n}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$, the LG-FIM is given by denoting $\tilde{\boldsymbol{\psi}}_G^{(i)} \overset{nota.}{=} \boldsymbol{\psi}_G\left(\boldsymbol{l}_G(\mathbf{X}_{cp}\mathbf{Y}_p^{(i)}, \mathbf{Z}^{(i)})\right)$ and $\mathbf{Ad}_G^{(i)} = \mathrm{Ad}_G\left({\mathbf{Y}_p^{(i)}}^{-1}\mathbf{X}_{cp}^{-1}\right)$, the adjoint representation of the LG G, we have:

$$J_1 = \sum_{i=1}^{N} \mathbb{E}\left({\tilde{\boldsymbol{\psi}}_G^{(i)}}^{\top} {\mathbf{Ad}_G^{(i)}}^{\top} \mathbf{\Sigma}^{-1}\mathbf{Ad}_G^{(i)}\, \tilde{\boldsymbol{\psi}}_G^{(i)}\right). \tag{40}$$

D. LG-MCRB and LG-CRB for a Gaussian random nuisance parameter

Solely for the purpose of comparing the proposed LG-MCRB with the LG-CRB, we also consider the case where the random nuisance parameter is modeled by,

$$\mathbf{Y}_p^{(i)} = \mathbf{I} \times \mathrm{Exp}^{\wedge}_{SO(2)}(b_i), \quad b_i \sim \mathcal{N}(0, \sigma_b^2). \tag{41}$$

Under this assumption, and $SO(2)$ being commutative,

$$\mathbf{Z}^{(i)} = \mathbf{X}_{cp}\mathrm{Exp}^{\wedge}_{SO(2)}(n_i + b_i), \quad i \in \{1, \dots, N\},$$

thus we are now able to derive the marginal pdf $p(\mathbf{Z}|\mathbf{X}_{cp})$:

$$p(\mathbf{Z}|\mathbf{X}_{cp}) \propto \exp\left(-\frac{1}{2}\sum_{i=1}^{N}\frac{l_{SO(2)}(\mathbf{X}_{cp}, \mathbf{Z}^{(i)})^2}{\sigma_n^2+\sigma_b^2}\right). \tag{42}$$

In order to compute the LG-MCRB, we provide the analytical expression of the joint pdf $p(\mathbf{Z}, \mathbf{Y}_p|\mathbf{X}_{cp})$:

$$p(\mathbf{Z}, \mathbf{Y}_p|\mathbf{X}_{cp}) \propto \exp\left(-\frac{1}{2}\sum_{i=1}^{N}\frac{l_{SO(2)}(\mathbf{X}_{cp}\mathbf{Y}_p^{(i)}, \mathbf{Z}^{(i)})^2}{\sigma_n^2}\right) \times \exp\left(-\frac{1}{2}\sum_{i=1}^{N}\frac{l_{SO(2)}(\mathbf{I}, \mathbf{Y}_p^{(i)})^2}{\sigma_b^2}\right). \tag{43}$$

Following similar lines as Sec. IV-C, while considering the likelihood (42) in the derivation steps of the LG-CRB, and the joint pdf (43) for the derivation of the LG-MCRB, we obtain the following expressions,

$$P_{\text{LG-CRB}} = \frac{\sigma_n^2+\sigma_b^2}{N} \geq P_{\text{LG-MCRB}} = \frac{\sigma_n^2}{N} \tag{44}$$

V. Numerical simulations

A. Simulation protocol and results

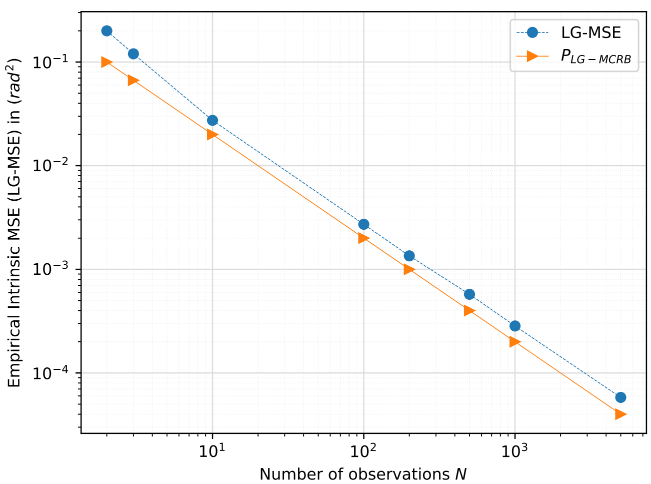
In this section, the proposed LG-MCRB is numerically validated by comparison with the LG-MSE of the estimator (29), considering the two priors defined in (34) and (41). To build the latter, the measurements are generated hierarchically: first drawing $\mathbf{Y}_p^{(i)}$ according to the considered prior, and then drawing $\mathbf{Z}^{(i)}$ according to (26) with a true value $\mathbf{X}_{cp} = \mathrm{Exp}^{\wedge}_{SO(2)}\left(\frac{\pi}{4}\right)$. The empirical intrinsic MSE (LG-MSE) is obtained as,

$$\text{LG-MSE} = \frac{1}{M_c}\sum_{m_c=1}^{M_c} l^2_{SO(2)}\left(\mathbf{X}_{cp}, \left(\widehat{\mathbf{X}}_{cp}\right)_{m_c}\right), \tag{45}$$

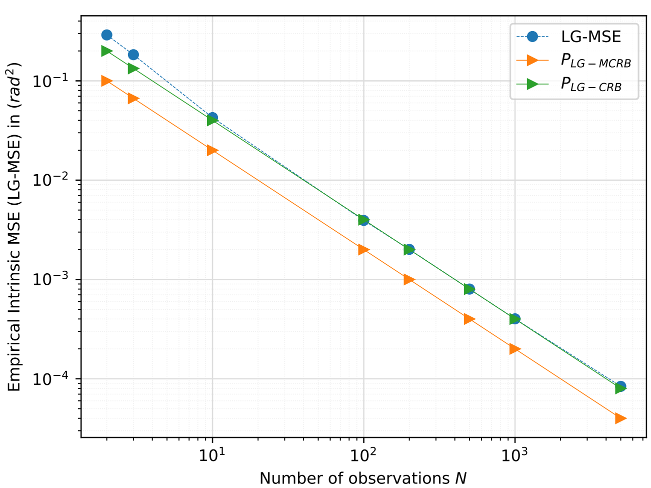
where M_c is the number of realizations ($M_c = 2000$ Monte Carlo trials) and $\left(\widehat{\mathbf{X}}_{cp}\right)_{m_c}$ is the estimator obtained, using the ML estimator (33), for the m_c-th realization of the observation noise. We carry out two computer experiments with fixed σ_n^2 and a varying number of observations:

- First prior: the random nuisance parameters are drawn according to $\mathbf{Y}_p^{(i)} = \mathbf{I}\times\mathrm{Exp}^{\wedge}_{SO(2)}(b_i), b_i \sim \mathcal{U}([-0.5, 0.5])$ (in rad) and the observations are drawn according to $\mathbf{Z}^{(i)} = \mathbf{X}_{cp}\mathbf{Y}_p^{(i)}\mathrm{Exp}^{\wedge}_{SO(2)}(n_i), n_i \sim \mathcal{N}(0, \sigma_n^2)$ with $\sigma_n^2 = 0.2\,\text{rad}^2$, for $1 \leq i \leq N$, where the total number of observations varies across the range $2 \leq N \leq 5000$. Fig. 2a displays the LG-MSE, alongside the LG-MCRB (39).

- Second prior: the random nuisance parameters are drawn according to $\mathbf{Y}_p^{(i)} = \mathbf{I} \times \text{Exp}^{\wedge}_{SO(2)}(b_i), b_i \sim \mathcal{N}(0, \sigma_b^2)$ with $\sigma_b^2 = 0.2\,\text{rad}^2$ and the observations are drawn according to $\mathbf{Z}^{(i)} = \mathbf{X}_{cp}\mathbf{Y}_p^{(i)}\text{Exp}^{\wedge}_{SO(2)}(n_i), n_i \sim \mathcal{N}(0, \sigma_n^2)$ with $\sigma_n^2 = 0.2\,\text{rad}^2$, for $1 \leq i \leq N$, where the total number of observations varies across the range $2 \leq N \leq 5000$. Fig. 2b displays the LG-MSE, alongside both LG-MCRB and LG-CRB (44).



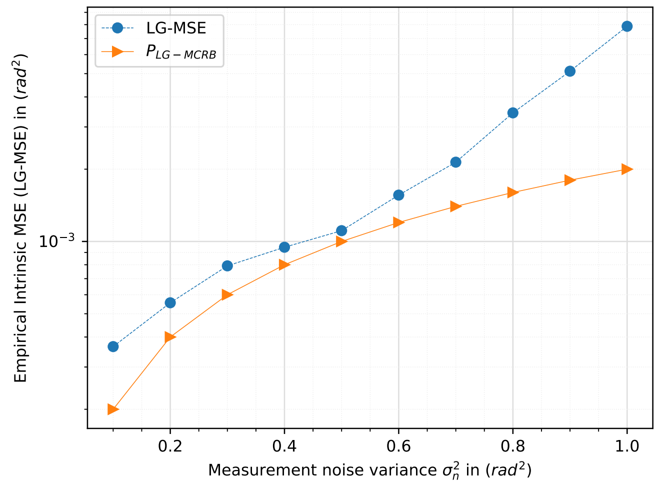
(a) Uniform prior



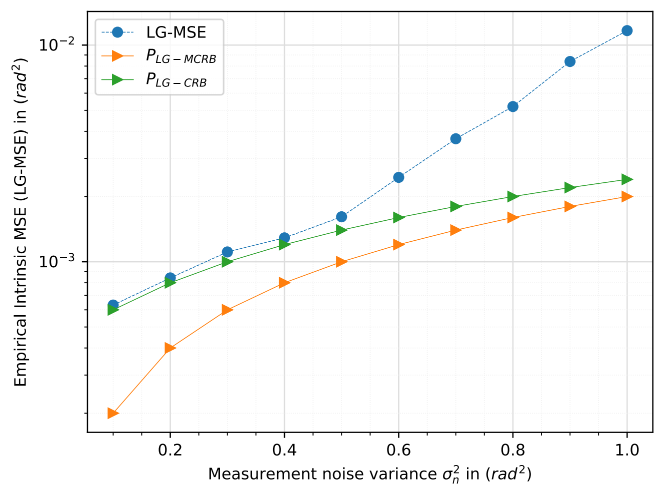
(b) Gaussian prior

Fig. 2: Evolution of the LG-MSE (blue dashed line), assuming a uniform prior in 2a, alongside the corresponding LG-MCRB (orange straight line) on $SO(2)$; and a Gaussian prior in 2b, alongside the corresponding LG-MCRB and LG-CRB (orange and green straight lines) on $SO(2)$; as the number of observations I increases ($2 \leq N \leq 5000$).

Furthermore, we carry out two similar computer experiments for a fixed number of observations $N = 500$ and a varying measurement noise variance $\sigma_n^2 \in [0.1, 1]$ (rad^2). The considered priors are the same as described in the previous experiments.



(a) Uniform prior



(b) Gaussian prior

Fig. 3: Evolution of the LG-MSE (blue dashed line), assuming a uniform prior in 3a, alongside the LG-MCRB (orange straight line) on $SO(2)$; and a Gaussian prior in 3b, alongside the LG-MCRB and LG-CRB (orange and green straight line); for $\sigma_n^2 \in [0.1, 1]$ (rad^2).

B. Discussion of results

The results in Fig. 2 and Fig. 3 numerically validate the proposed LG-MCRB. The latter consistently lower bounds the LG-MSE, across varying measurement noise variance values σ_n^2, as illustrated in Fig. 3, and varying number of observations N, as illustrated in Fig. 2. These results hold irrespective of the prior on the random nuisance parameter. Similar to the well-known Euclidean results, the LG-CRB and LG-MCRB on $SO(2)$ are valid when the signal-to-noise ratio (SNR) is high or the number of observations is large. For low SNR, the LG-MSE rises sharply due to the increase in bias. Furthermore, in line with the discussions on the comparison between CRB and MCRB in the Euclidean case [5], [6], while the MCRB guarantees general feasibility, it is indeed slightly looser than the standard CRB, as highlighted by inequality (44) and further illustrated by Fig. 2b and Fig. 3b. However, the LG-MCRB is relevant in situations where the LG-CRB is inaccessible, and we observe in Fig. 2a that the gap between LG-MSE and LG-MCRB is small. We also observe in Fig. 3b that the LG-MCRB approaches the LG-CRB as the measurement uncertainty σ_n^2 increases, which could suggest that the LG-MCRB is tighter in situations involving larger levels of measurement noise.

VI. Conclusion

In this article, we derived a generalization of the MCRB on LGs, which proves invaluable in non-standard estimation problems. By adapting the intrinsic unbiasedness constraint and the Barankin Bound formalism to this class of estimation problems, we were able to derive the sought LG-MCRB. We further derived closed-form expressions of the latter for a Gaussian model on $SO(2)$ with parameters on $SO(2)$, in the presence of random nuisance parameters. The findings through numerical simulations underscore the validity and relevance of the proposed bound to address non-standard estimation problems. Future work will address more in depth the current findings on the slight discrepancy between LG-CRB and LG-MCRB. In fact, as detailed in [7, §III.B], authors proposed a more stringent class of MCRBs in the Euclidean framework. Nonetheless, the constraints are associated with the Hybrid CRB, and their extension to LGs is not straightforward.

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