# Cooperative positioning using pseudorange measurements: solvability and conservative algorithms

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TéSA, January 30, 2025

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solvability and conservative algorithms

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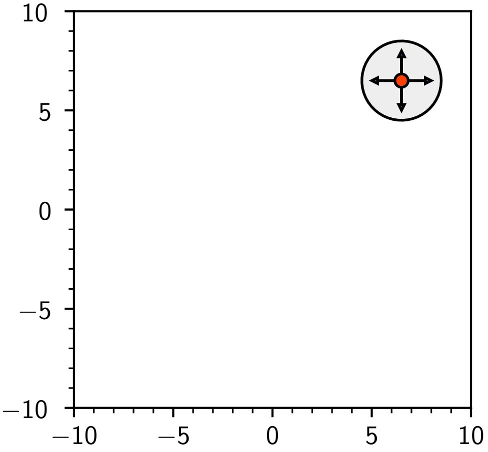
# Positioning of a user

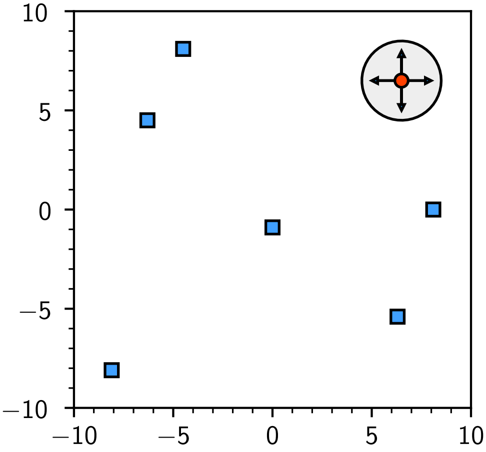
# Positioning of a user

1. A reference frame



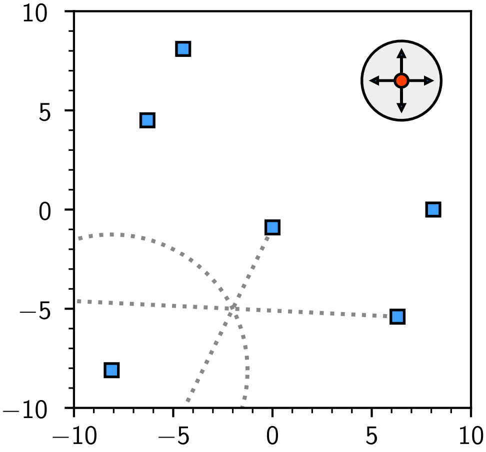
# Positioning of a user

1. A reference frame
2. Reference points (Anchors ◼)



# Positioning of a user

1. A reference frame
2. Reference points (Anchors ■)
3. Measurements (···)



# Positioning of a user

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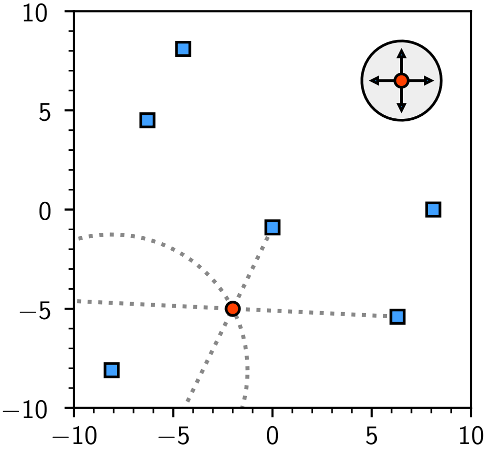
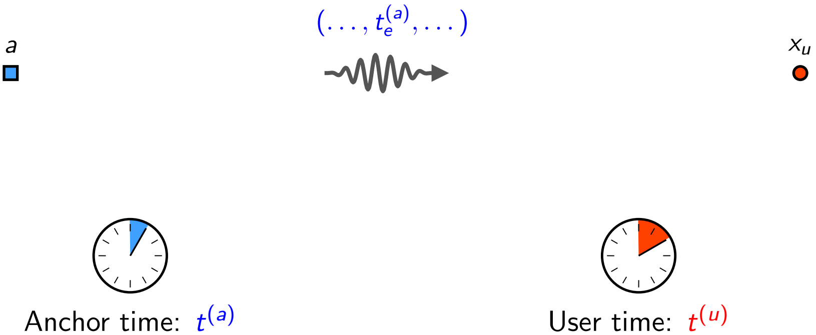
⇒ User's position (●)



# Pseudorange

$$\rho_{a \to u} \triangleq c \overbrace{\left( t_r^{(u)} - t_e^{(a)} \right)}^{\text{t.o.f.}}$$



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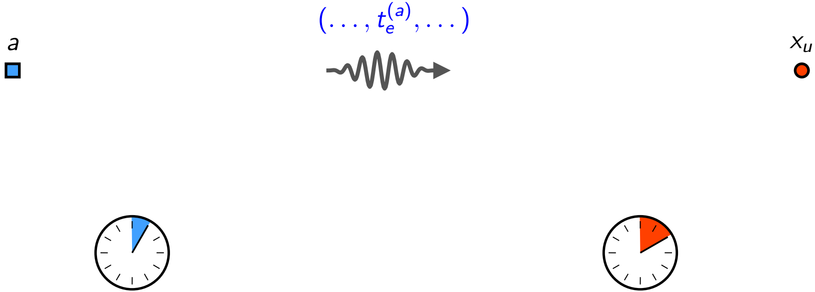
$$\rho_{a\to u} \triangleq c\,\overbrace{\left(t_r^{(u)} - t_e^{(a)}\right)}^{\text{t.o.f.}}$$

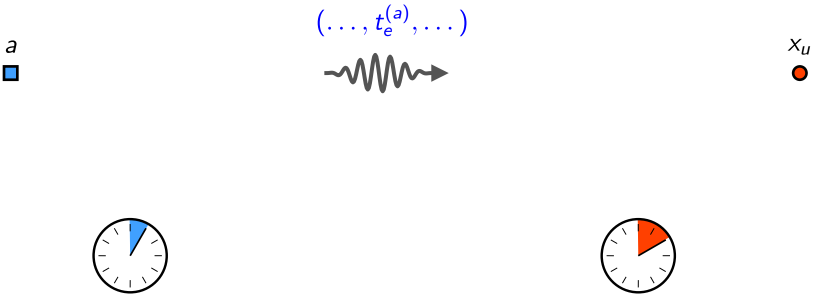



Anchor time: $t^{(a)} = t + \tau_a$

User time: $t^{(u)} = t + \tau_u$

# Pseudorange

$$\rho_{a\to u} \triangleq c\overbrace{\left(t_r^{(u)} - t_e^{(a)}\right)}^{\text{t.o.f.}} = c\,(t_r - t_e) + c\,(\tau_u - \tau_a)$$
$$= \|a - x_u\| + \beta_u - \beta_a$$

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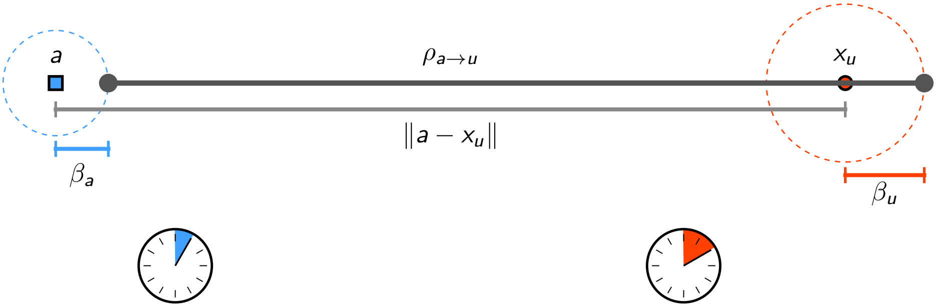
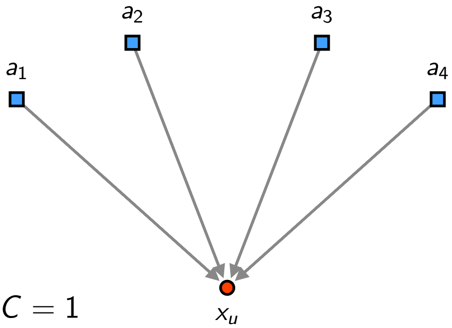



Anchor time: $t^{(a)} = t + \tau_a$

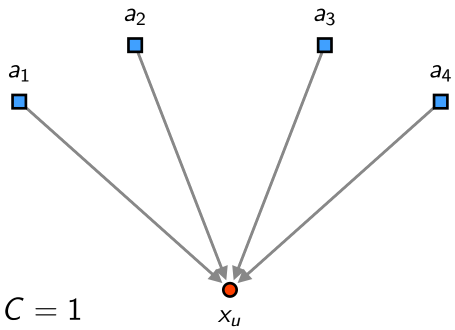
User time: $t^{(u)} = t + \tau_u$

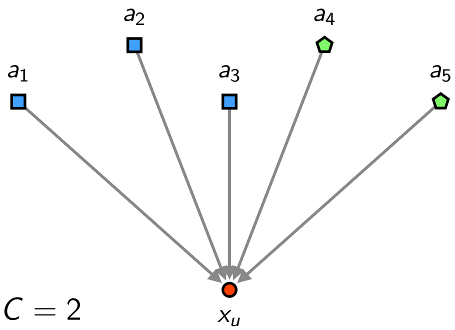
# GNSS positioning with $C$ constellations in $\mathbb{R}^d$



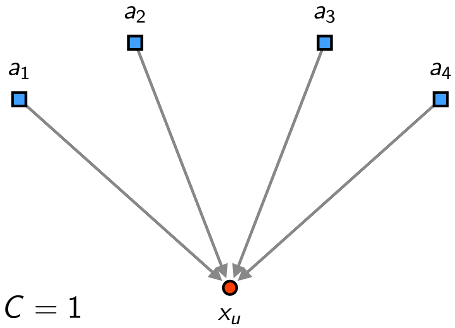

$C = 1$

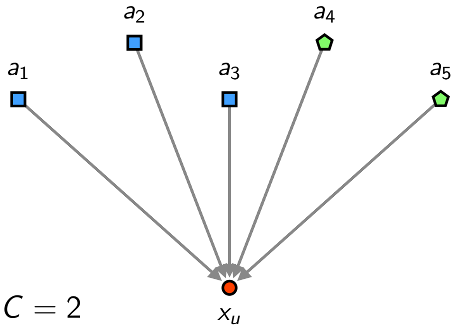
# GNSS positioning with $C$ constellations in $\mathbb{R}^d$



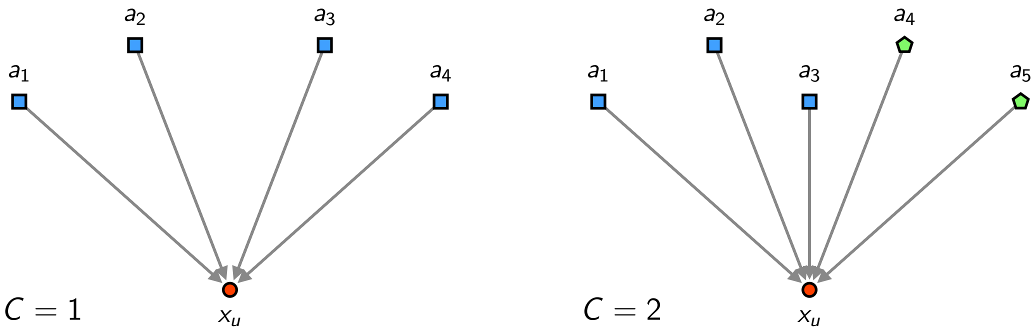
# GNSS positioning with $C$ constellations in $\mathbb{R}^d$




$3 + C$ measurements to solve the problem in $\mathbb{R}^3$

# GNSS positioning with $C$ constellations in $\mathbb{R}^d$



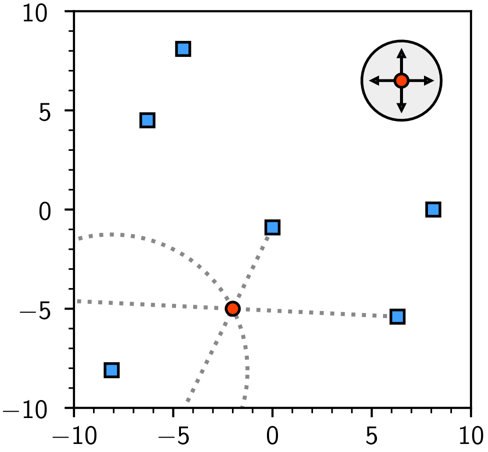

$3 + C$ measurements to solve the problem in $\mathbb{R}^3$

$d + C$ measurements to solve the problem in $\mathbb{R}^d$

# Cooperative positioning

1. A reference frame
2. Reference points (Anchors ■)
3. Measurements (···)

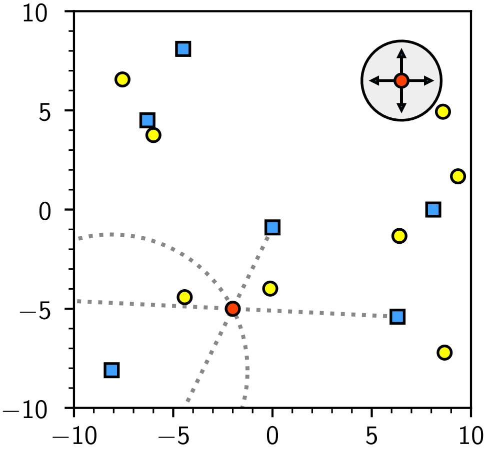
   ⇒ User's position (●)



# Cooperative positioning

1. A reference frame
2. Reference points (Anchors ■)
3. Measurements (···)

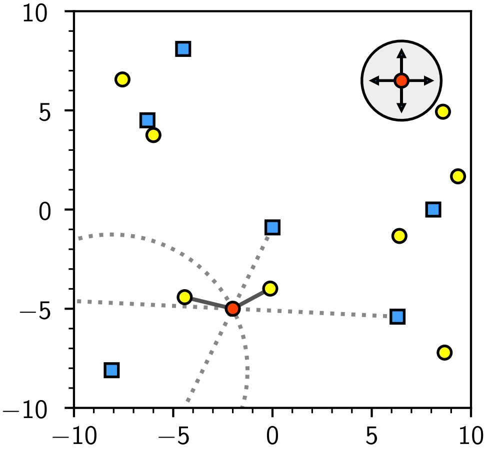
   ⇒ User's position (●)



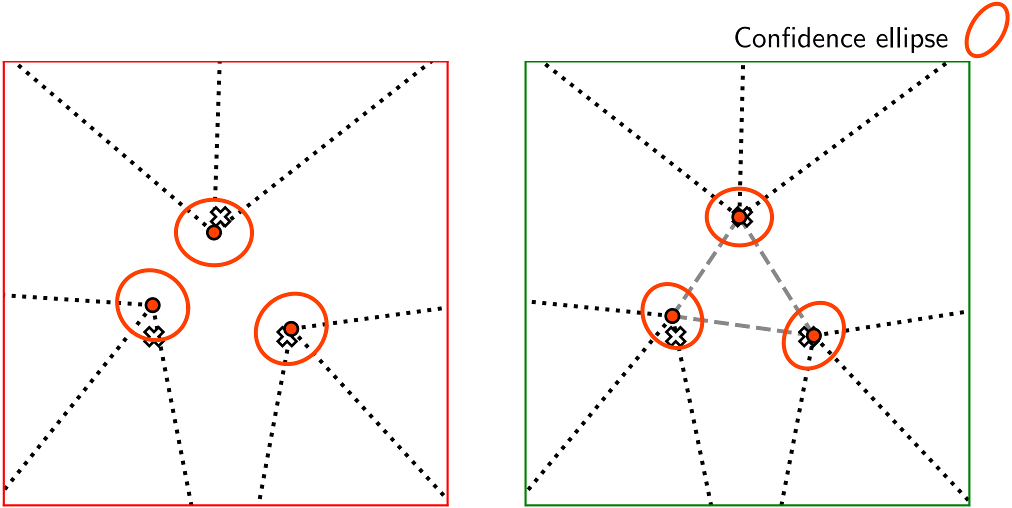
# Cooperative positioning

1. A reference frame
2. Reference points:
   - Anchors ■
   - Other users ●
3. Measurements (···)

⇒ User's position ( ● )



# Cooperative positioning

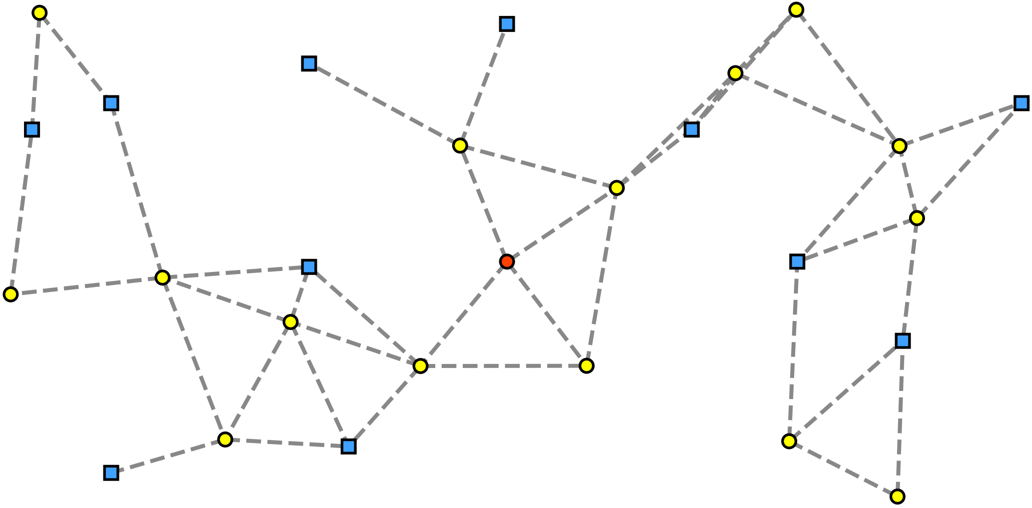


# Challenge 1: Is the problem solvable?

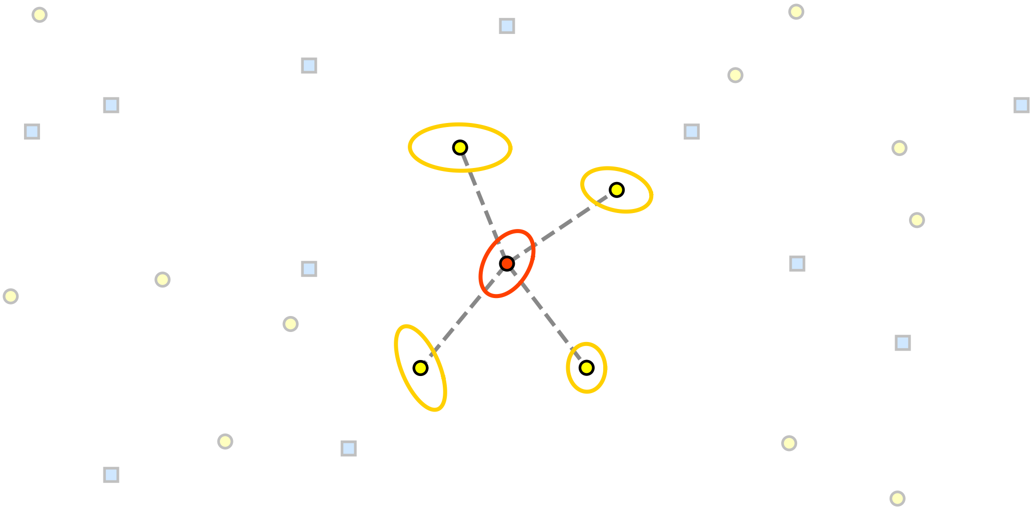
**A single user:**

$d + C$ measurements to solve the problem

# Challenge 1: Is the problem solvable?



# Challenge 2: How to use the cooperative measurements?



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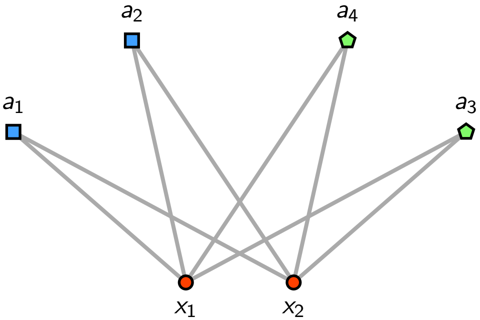
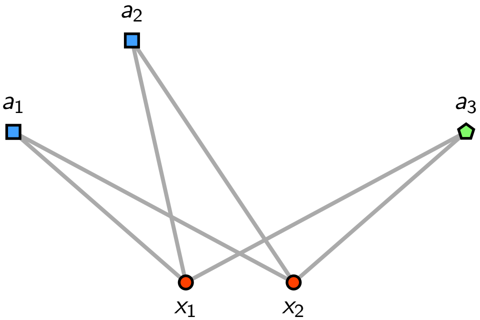
# Example in 2D



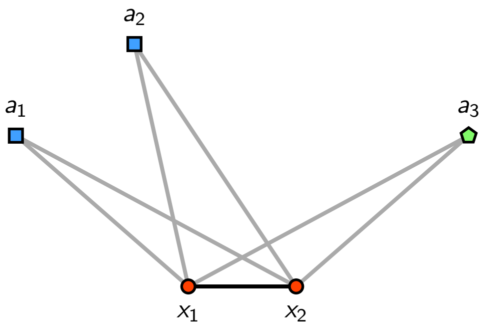

2+C: Solvable

# Example in 2D without 1 satellite



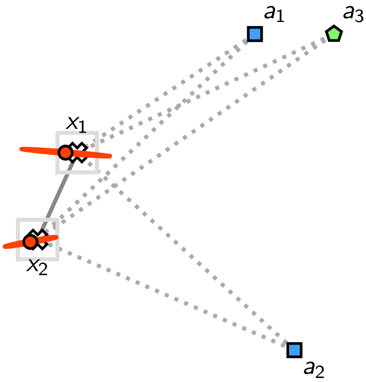

1+C: Unsolvable

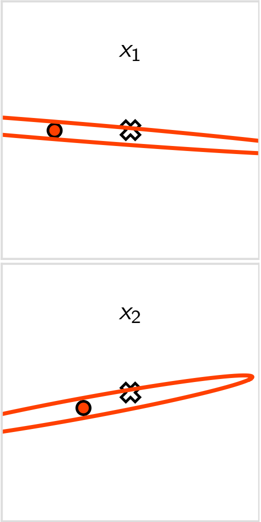
# Example in 2D without 1 satellite and with cooperation



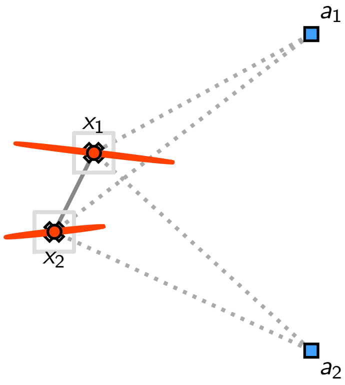

1+C+1: (Un)solvable?

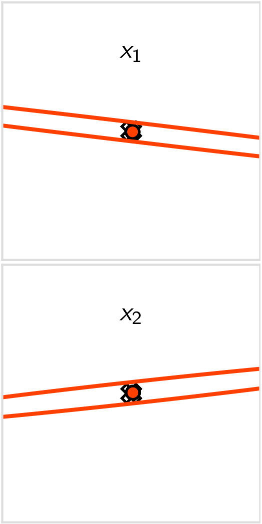
# Solvability test in bi-constellation ($C = 2$)



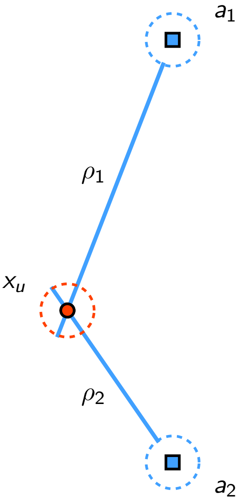



# Solvability test in mono-constellation ($C = 1$)



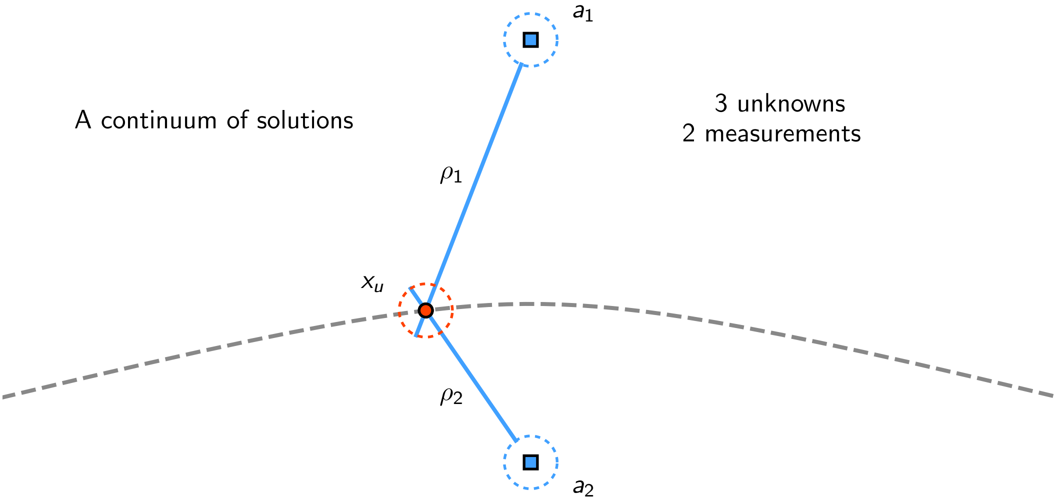
# Pseudorange constraints with two anchors




3 unknowns
2 measurements

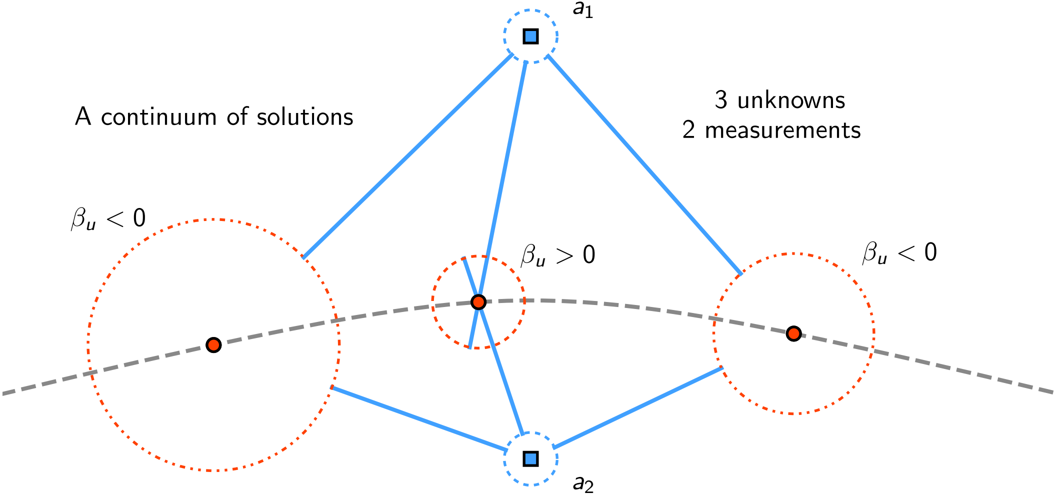
# Pseudorange constraints with two anchors

A continuum of solutions




3 unknowns
2 measurements

# Pseudorange constraints with two anchors



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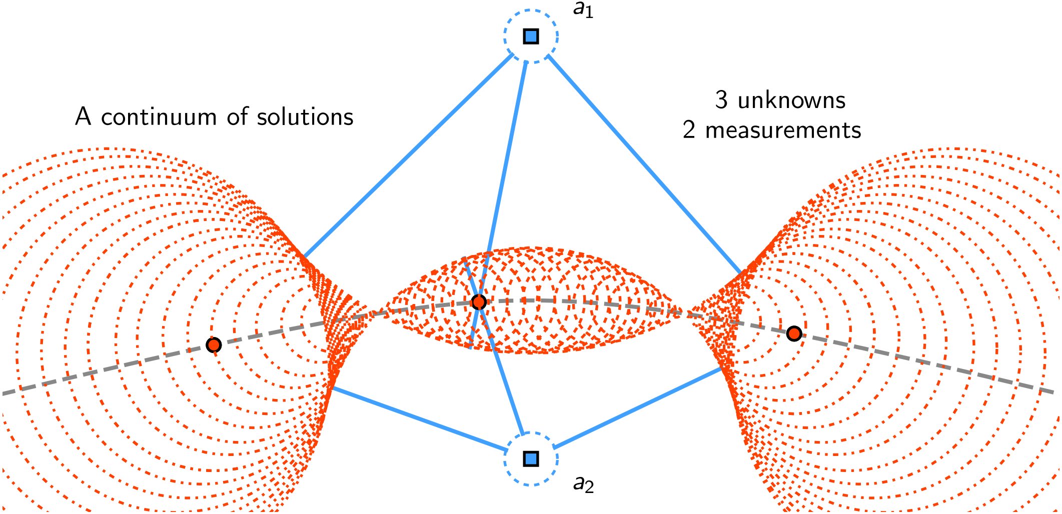
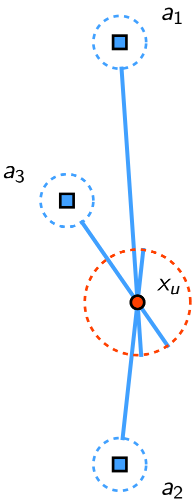
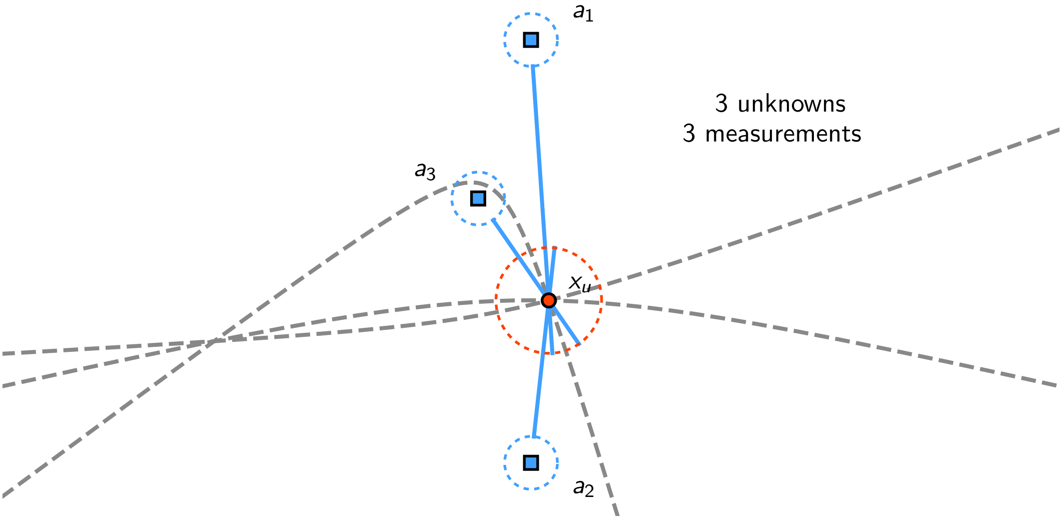


# Pseudorange constraints with three anchors




3 unknowns
3 measurements

# Pseudorange constraints with three anchors




3 unknowns
3 measurements

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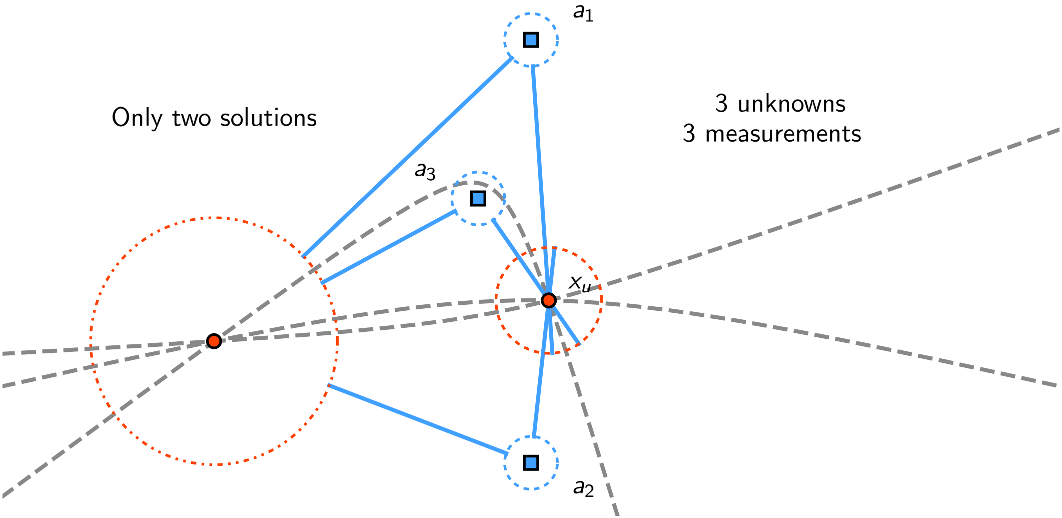
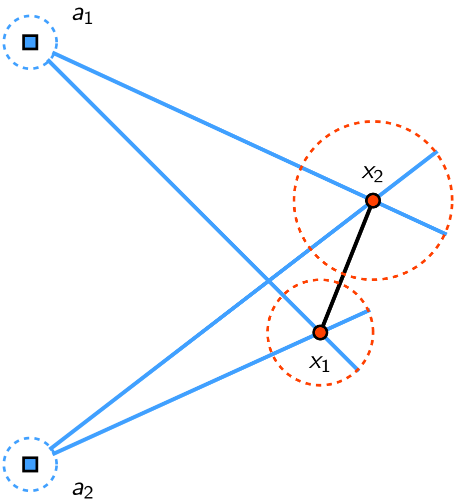
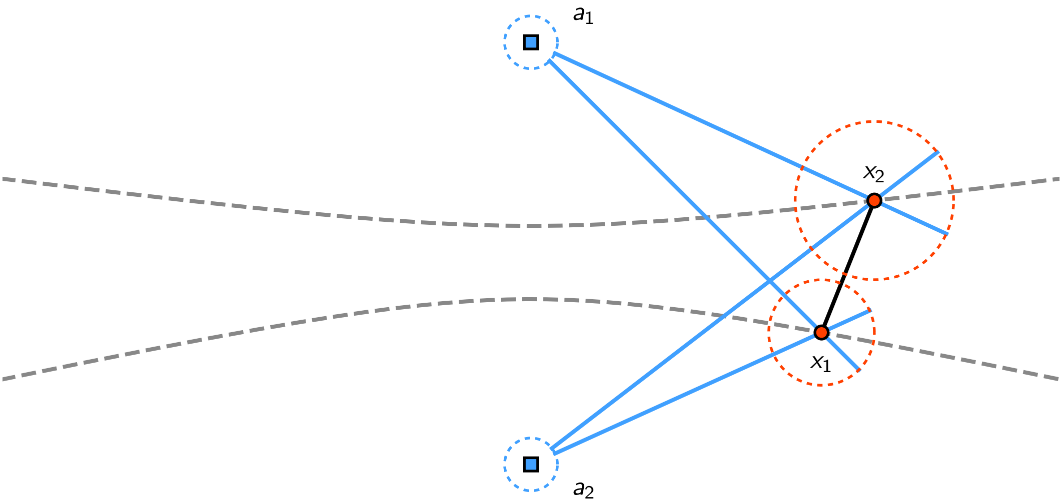


# Cooperative positioning



# Cooperative positioning



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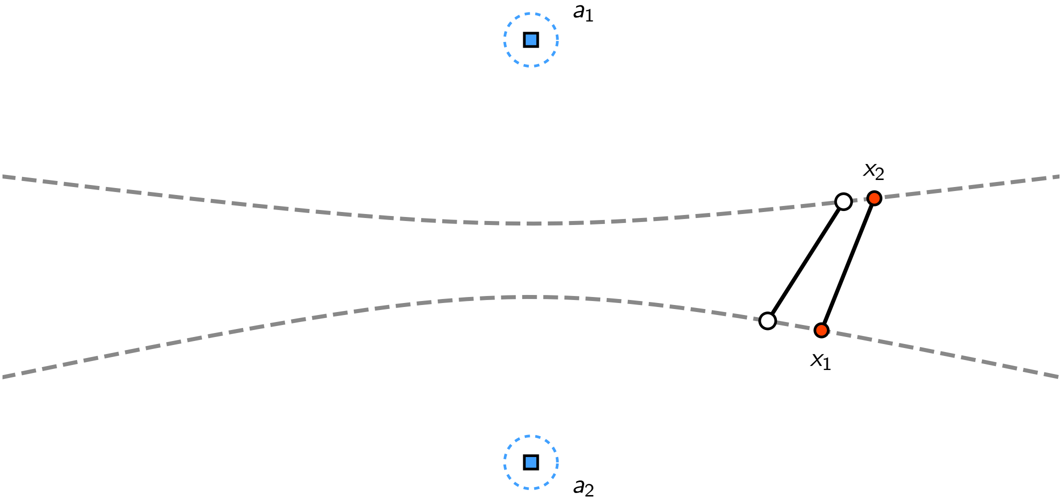
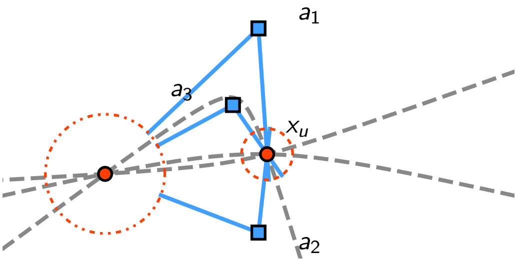
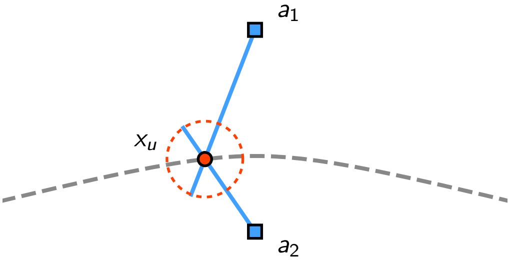


# To sum up: Solvable ⇔ Discrete solution set

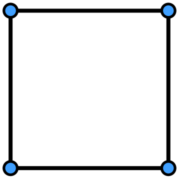



Solvable



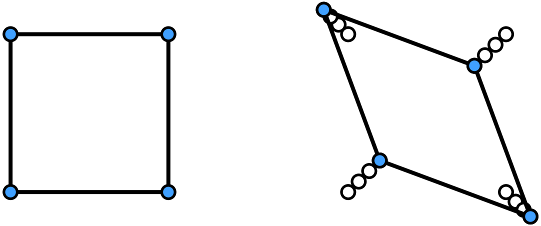

Unsolvable

# Rigidity



Bill Jackson. "Notes on the rigidity of graphs". In: *Levico Conference Notes*. Vol. 4. 2007

# Rigidity



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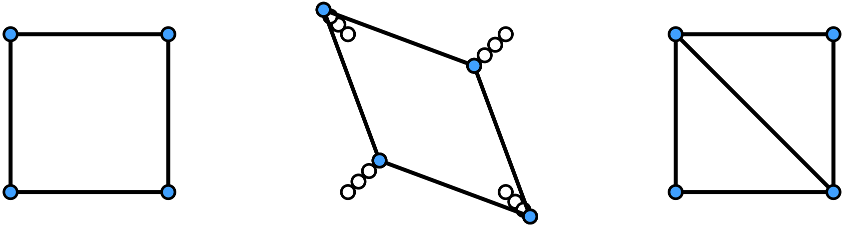
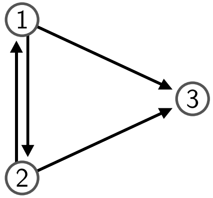
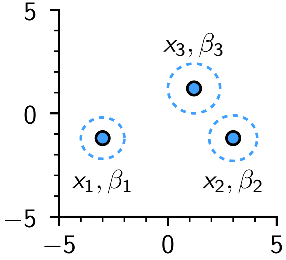


Bill Jackson. "Notes on the rigidity of graphs". In: *Levico Conference Notes*. Vol. 4. 2007

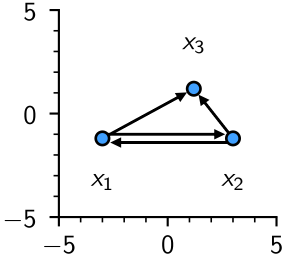
# Pseudorange frameworks




Graph of constraints Γ




Configuration $\boldsymbol{p}$

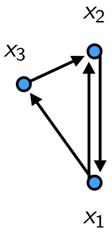


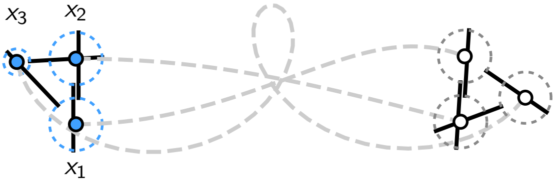

Framework $(\Gamma, \boldsymbol{p})$

Pseudorange constraint: $$f_P(\boldsymbol{p}, uw) \triangleq \|\boldsymbol{x}_u - \boldsymbol{x}_w\| + \beta_w - \beta_u = \text{CST}.$$

# Pseudorange infinitesimal rigidity (1/2)

$$f_P(\boldsymbol{p}, uw) \triangleq \|\boldsymbol{x}_u - \boldsymbol{x}_w\| + \beta_w - \beta_u = \text{CST}$$



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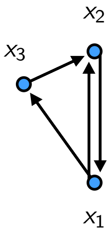


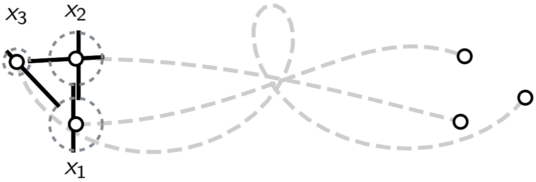

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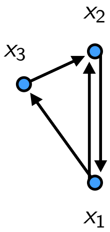
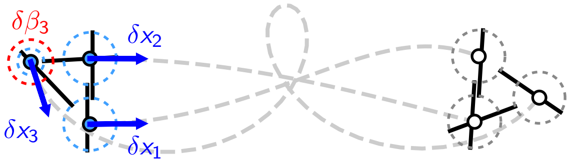
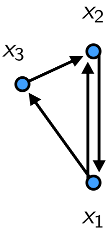
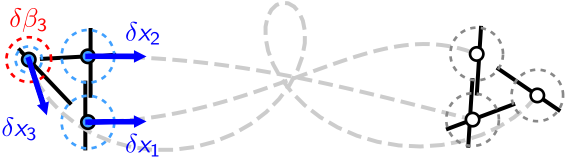


# Pseudorange infinitesimal rigidity (1/2)

$$f_P(\boldsymbol{p}, uw) \triangleq \|\boldsymbol{x}_u - \boldsymbol{x}_w\| + \beta_w - \beta_u = \text{CST} \qquad \nabla_{\boldsymbol{p}} f_P(\boldsymbol{p}, uw)^\intercal \cdot \begin{pmatrix} \boldsymbol{\delta x} \\ \boldsymbol{\delta \beta} \end{pmatrix} = 0$$

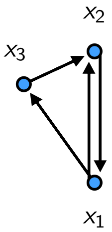


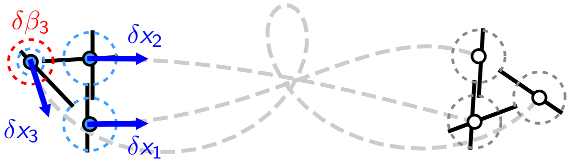



# Pseudorange infinitesimal rigidity (1/2)

$$f_P(\boldsymbol{p}, uw) \triangleq \|\boldsymbol{x}_u - \boldsymbol{x}_w\| + \beta_w - \beta_u = \text{CST} \qquad \nabla_{\boldsymbol{p}} f_P(\boldsymbol{p}, uw)^\intercal \cdot \begin{pmatrix} \boldsymbol{\delta x} \\ \boldsymbol{\delta \beta} \end{pmatrix} = 0$$

$$\Rightarrow \qquad (\boldsymbol{x}_u - \boldsymbol{x}_w)^\intercal (\boldsymbol{\delta x}_u - \boldsymbol{\delta x}_w) + \|\boldsymbol{x}_u - \boldsymbol{x}_w\| (\delta\beta_w - \delta\beta_u) = 0$$

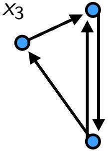


# Pseudorange infinitesimal rigidity (2/2)

$$f_P(\boldsymbol{p}, uw) \triangleq \|\boldsymbol{x}_u - \boldsymbol{x}_w\| + \beta_w - \beta_u = \text{CST}$$

$$(\boldsymbol{x}_u - \boldsymbol{x}_w)^\intercal (\boldsymbol{\delta x}_u - \boldsymbol{\delta x}_w) + \|\boldsymbol{x}_u - \boldsymbol{x}_w\| (\delta\beta_w - \delta\beta_u) = 0$$




$$\boldsymbol{R}_P(\Gamma, \boldsymbol{p}) = \left[\begin{array}{ccc|ccc} \boldsymbol{x}_1^\intercal - \boldsymbol{x}_2^\intercal & \boldsymbol{x}_2^\intercal - \boldsymbol{x}_1^\intercal & \boldsymbol{0}^\intercal & -d_{12} & d_{12} & 0 \\ \boldsymbol{x}_1^\intercal - \boldsymbol{x}_2^\intercal & \boldsymbol{x}_2^\intercal - \boldsymbol{x}_1^\intercal & \boldsymbol{0}^\intercal & d_{12} & -d_{12} & 0 \\ \boldsymbol{x}_1^\intercal - \boldsymbol{x}_3^\intercal & \boldsymbol{0}^\intercal & \boldsymbol{x}_3^\intercal - \boldsymbol{x}_1^\intercal & -d_{13} & 0 & d_{13} \\ \boldsymbol{0}^\intercal & \boldsymbol{x}_2^\intercal - \boldsymbol{x}_3^\intercal & \boldsymbol{x}_3^\intercal - \boldsymbol{x}_2^\intercal & 0 & -d_{23} & d_{23} \end{array}\right]$$

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# Pseudorange infinitesimal rigidity (2/2)

$$f_P(\boldsymbol{p}, uw) \triangleq \|\boldsymbol{x}_u - \boldsymbol{x}_w\| + \beta_w - \beta_u = \text{CST}$$

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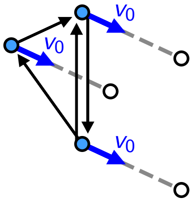

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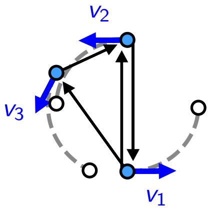

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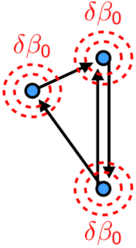


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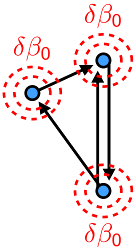


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$$S_P(N, d) \triangleq S_D(N, d) + N - 1$$

Leonard Asimow and Ben Roth. "The rigidity of graphs". In: *Transactions of the American Mathematical Society* 245 (1978), pp. 279–289

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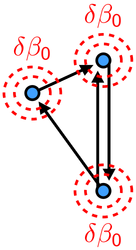


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**Definition:**

$(\Gamma, \boldsymbol{p})$ is infinitesimally rigid if rank $\boldsymbol{R}_P(\Gamma, \boldsymbol{p}) = S_P(N, d)$.

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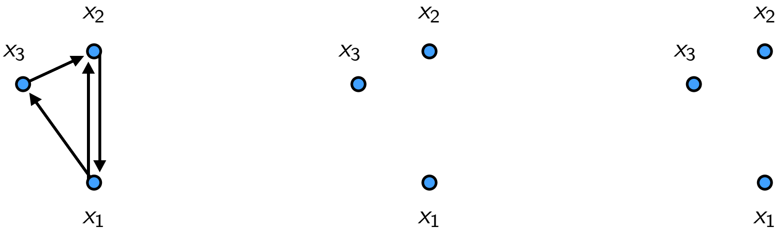
Leonard Asimow and Ben Roth. "The rigidity of graphs". In: *Transactions of the American Mathematical Society* 245 (1978), pp. 279–289

# Decompositions of the pseudorange graph

$$\boldsymbol{R}_P(\Gamma, \boldsymbol{p}) = \left[\begin{array}{ccc|ccc} \boldsymbol{x}_1^\top - \boldsymbol{x}_2^\top & \boldsymbol{x}_2^\top - \boldsymbol{x}_1^\top & \boldsymbol{0}^\top & -d_{12} & d_{12} & 0 \\ \boldsymbol{x}_1^\top - \boldsymbol{x}_2^\top & \boldsymbol{x}_2^\top - \boldsymbol{x}_1^\top & \boldsymbol{0}^\top & d_{12} & -d_{12} & 0 \\ \boldsymbol{x}_1^\top - \boldsymbol{x}_3^\top & \boldsymbol{0}^\top & \boldsymbol{x}_3^\top - \boldsymbol{x}_1^\top & -d_{13} & 0 & d_{13} \\ \boldsymbol{0}^\top & \boldsymbol{x}_2^\top - \boldsymbol{x}_3^\top & \boldsymbol{x}_3^\top - \boldsymbol{x}_2^\top & 0 & -d_{23} & d_{23} \end{array}\right]$$




$\Gamma$ $\qquad$ $G_D$ $\qquad$ $G_S$

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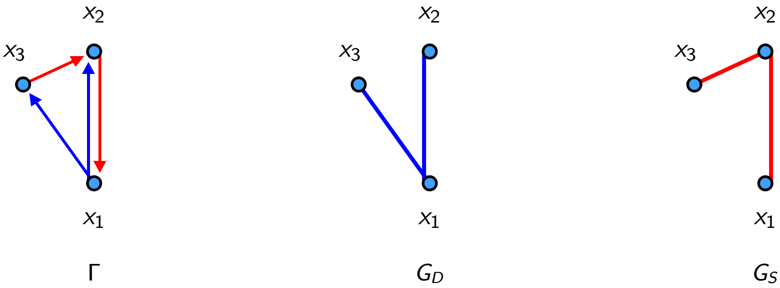

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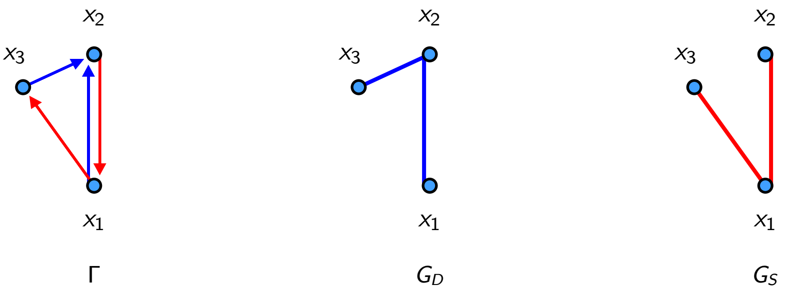

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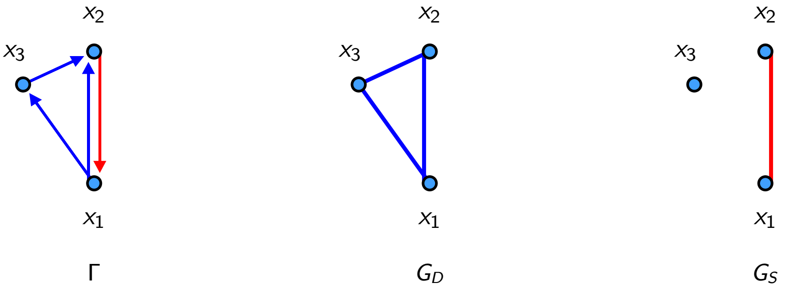

# Main result

$$\boldsymbol{R}_P(\Gamma, p) \sim \begin{bmatrix} \boldsymbol{R}_D(G_D, \boldsymbol{p}) & \boldsymbol{R}_S(G_D, \boldsymbol{p}) \\ \boldsymbol{R}_D(G_S, \boldsymbol{p}) & \boldsymbol{R}_S(G_S, \boldsymbol{p}) \end{bmatrix}$$

**Theorem [C1]:**

$$\operatorname{rank} \boldsymbol{R}_P(\Gamma, \boldsymbol{p}) = \max_{G_D \cup G_S = \Gamma} \operatorname{rank} \boldsymbol{R}_D(G_D, \boldsymbol{p}) + \operatorname{rank} \boldsymbol{R}_S(G_S, \boldsymbol{p})$$

[C1] Colin Cros et al. "Pseudorange Rigidity and Solvability of Cooperative GNSS Positioning". In: *IEEE Transactions on Control of Network Systems* (2024), pp. 1–12

# Corollaries

$$\boldsymbol{R}_P(\Gamma, \boldsymbol{p}) = \left[\begin{array}{ccc|ccc} \boldsymbol{x}_1^\top - \boldsymbol{x}_2^\top & \boldsymbol{x}_2^\top - \boldsymbol{x}_1^\top & \boldsymbol{0}^\top & -d_{12} & d_{12} & 0 \\ \boldsymbol{x}_1^\top - \boldsymbol{x}_2^\top & \boldsymbol{x}_2^\top - \boldsymbol{x}_1^\top & \boldsymbol{0}^\top & d_{12} & -d_{12} & 0 \\ \boldsymbol{x}_1^\top - \boldsymbol{x}_3^\top & \boldsymbol{0}^\top & \boldsymbol{x}_3^\top - \boldsymbol{x}_1^\top & -d_{13} & 0 & d_{13} \\ \boldsymbol{0}^\top & \boldsymbol{x}_2^\top - \boldsymbol{x}_3^\top & \boldsymbol{x}_3^\top - \boldsymbol{x}_2^\top & 0 & -d_{23} & d_{23} \end{array}\right]$$

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**Corollary 1 [C1]:**

$(\Gamma, \boldsymbol{p})$ Pseudorange-Rigid is a generic property of $\Gamma$.

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$$\operatorname{rank} R_P(\Gamma, \boldsymbol{p}) = \max_{G_D \cup G_S = \Gamma} \operatorname{rank} R_D(G_D, \boldsymbol{p}) + \operatorname{rank} R_S(G_S, \boldsymbol{p})$$

**Corollary 1 [C1]:**

$(\Gamma, \boldsymbol{p})$ Pseudorange-Rigid is a generic property of $\Gamma$.

**Corollary 2 [C1]:**

$\Gamma$ Pseudorange-Rigid iff there exists a decomposition $(G_D, G_S)$ with $G_D$ Distance-Rigid and $G_S$ connected.

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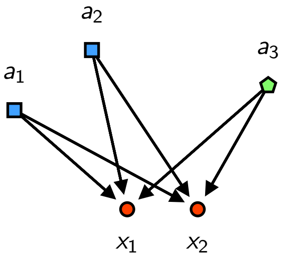
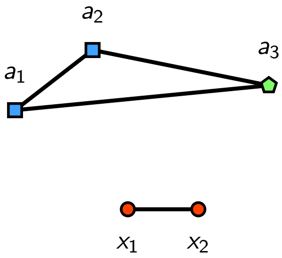
# Application with the initial example

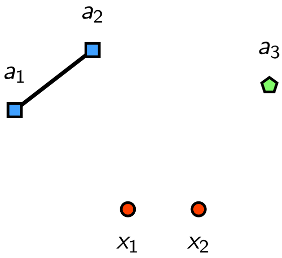
**Three types of constraints:**




Pseudorange constr.



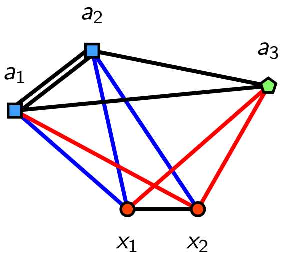

Distance constr.



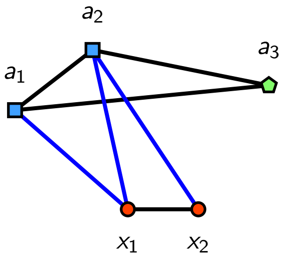

Synchronization constr.

# Application with the initial example

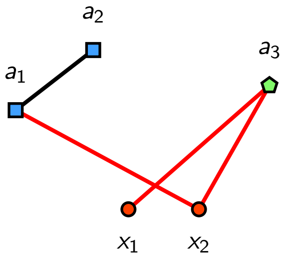
**Decomposition:**




Constraints



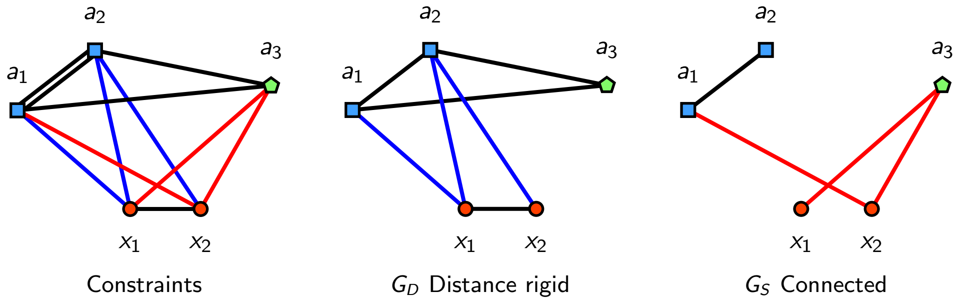

$G_D$ Distance rigid




$G_S$ Connected

# Application with the initial example

**Decomposition:**




Distance Rigid Graph + Connected Graph $\Rightarrow$ Solvable Problem

# Take home message

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1. To be able to track a network in the GNSS context, the graph of measurements must be rigid. It must set the positions of the agents and synchronize their clocks.

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1. To be able to track a network in the GNSS context, the graph of measurements must be rigid. It must set the positions of the agents and synchronize their clocks.

2. A new rigidity theory adapted to the pseudorange measurements has been developed.

3. Result: A pseudorange graph is rigid if and only if it can be decomposed into a distance rigid graph and connected graph.

# Table of contents

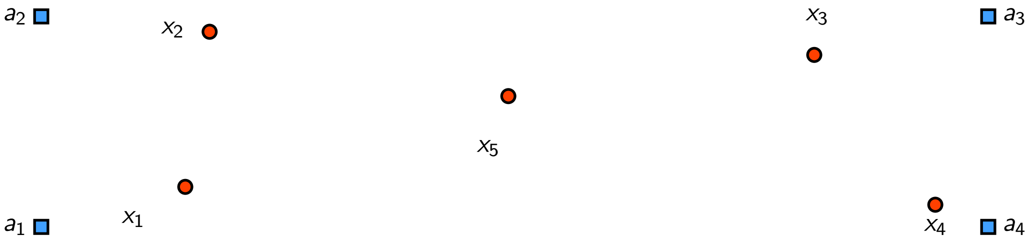
# Linearized model

Dynamics

$$\boldsymbol{x}_i(k+1) = \boldsymbol{F}_i \boldsymbol{x}_i(k) + \boldsymbol{w}_i(k)$$

Measurements

$$z_i(k) = \left( \boldsymbol{z}_i^{(auto)}(k)^\mathsf{T} \quad \boldsymbol{z}_i^{(coop)}(k)^\mathsf{T} \right)^\mathsf{T}$$



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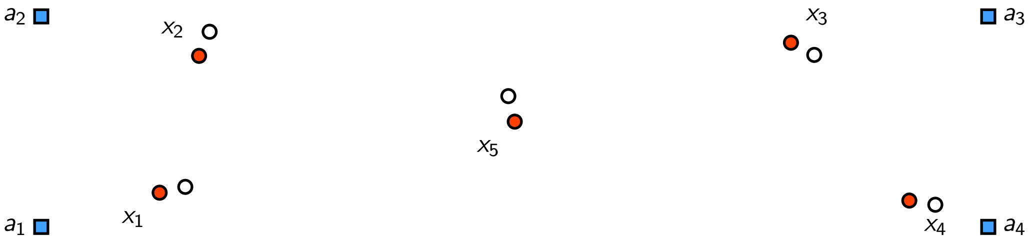
Measurements

$$z_i(k) = \left( \boldsymbol{z}_i^{(auto)}(k)^\intercal \quad \boldsymbol{z}_i^{(coop)}(k)^\intercal \right)^\intercal$$



# Linearized model

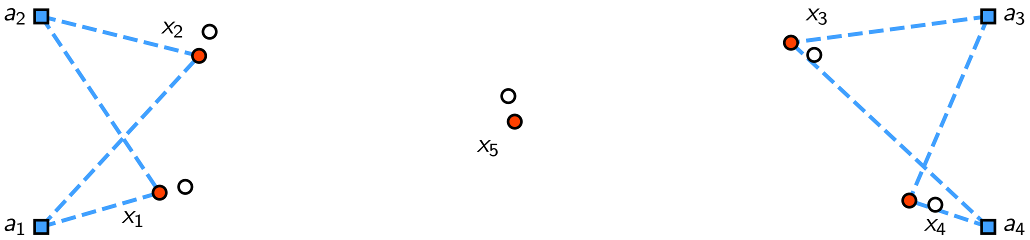
Dynamics

$$\boldsymbol{x}_i(k+1) = \boldsymbol{F}_i \boldsymbol{x}_i(k) + \boldsymbol{w}_i(k)$$

Measurements

$$z_i(k) = \left( \boldsymbol{z}_i^{(auto)}(k)^\intercal \quad \boldsymbol{z}_i^{(coop)}(k)^\intercal \right)^\intercal$$

$$z_i^{(auto)}(k) = \boldsymbol{H}_i^{(auto)} \boldsymbol{x}_i^{(auto)}(k) + \boldsymbol{v}_i^{(auto)}(k)$$



# Linearized model

Dynamics

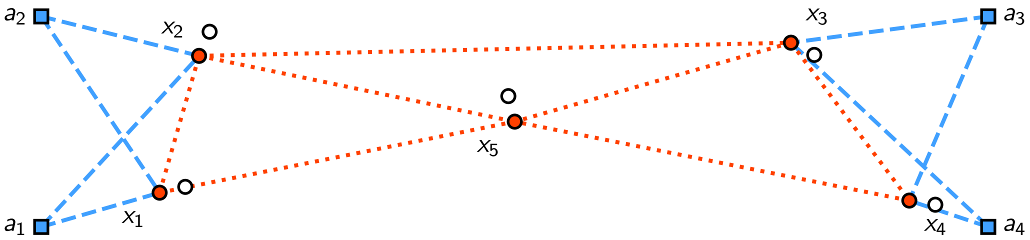
$$\boldsymbol{x}_i(k+1) = \boldsymbol{F}_i \boldsymbol{x}_i(k) + \boldsymbol{w}_i(k)$$

Measurements

$$z_i(k) = \left( \boldsymbol{z}_i^{(auto)}(k)^\intercal \quad \boldsymbol{z}_i^{(coop)}(k)^\intercal \right)^\intercal$$

$$z_i^{(auto)}(k) = \boldsymbol{H}_i^{(auto)} \boldsymbol{x}_i^{(auto)}(k) + \boldsymbol{v}_i^{(auto)}(k)$$

$$z_i^{(coop)}(k) = \sum_j \boldsymbol{H}_{i,j}^{(coop)} \boldsymbol{x}_j^{(coop)}(k) + \boldsymbol{v}_i^{(coop)}(k)$$



# Linearized model

Dynamics

$$\boldsymbol{x}_i(k+1) = \boldsymbol{F}_i \boldsymbol{x}_i(k) + \boldsymbol{w}_i(k)$$

$\hat{\boldsymbol{x}}_i$ : estimation, $\quad \tilde{\boldsymbol{x}}_i \triangleq \hat{\boldsymbol{x}}_i - \boldsymbol{x}_i,$

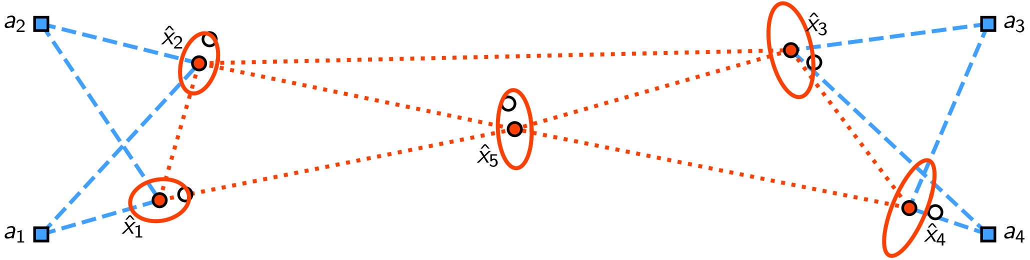
$\tilde{\boldsymbol{P}}_i \triangleq \mathrm{E}[\tilde{\boldsymbol{x}}_i \tilde{\boldsymbol{x}}_i^\mathsf{T}]$

Measurements

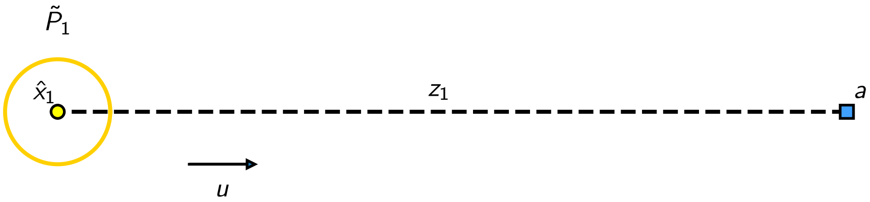
$$z_i(k) = \begin{pmatrix} \boldsymbol{z}_i^{(auto)}(k)^\mathsf{T} & \boldsymbol{z}_i^{(coop)}(k)^\mathsf{T} \end{pmatrix}^\mathsf{T}$$

$$z_i^{(auto)}(k) = \boldsymbol{H}_i^{(auto)} \boldsymbol{x}_i^{(auto)}(k) + \boldsymbol{v}_i^{(auto)}(k)$$

$$z_i^{(coop)}(k) = \sum_j \boldsymbol{H}_{i,j}^{(coop)} \boldsymbol{x}_j^{(coop)}(k) + \boldsymbol{v}_i^{(coop)}(k)$$

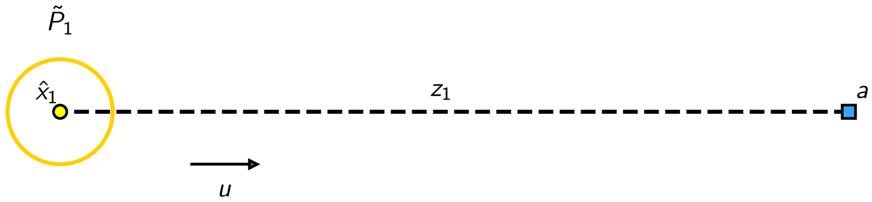


# Filtering of an autonomous measurement




$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left(z_1 - \boldsymbol{u}^\top \hat{\boldsymbol{x}}_1\right)$$

# Filtering of an autonomous measurement

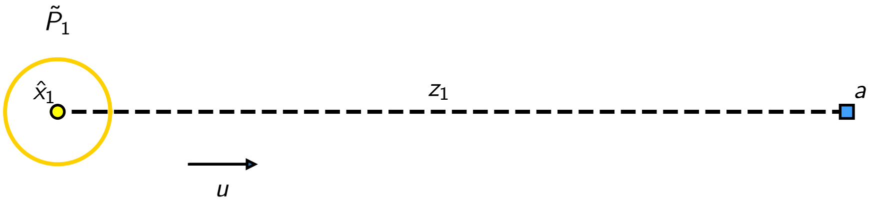



$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left(z_1 - \boldsymbol{u}^\top \hat{\boldsymbol{x}}_1\right)$$

$$\boldsymbol{K} = \frac{\tilde{\boldsymbol{P}}_1 \boldsymbol{u}}{\sigma_1^2 + \sigma_m^2}$$

$$\sigma_1^2 = \boldsymbol{u}^\top \tilde{\boldsymbol{P}}_1 \boldsymbol{u}$$

# Filtering of an autonomous measurement



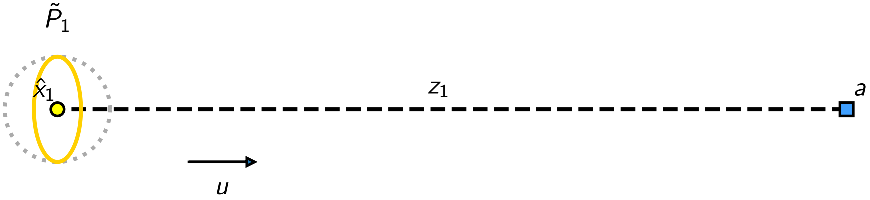

$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left(z_1 - \boldsymbol{u}^\top \hat{\boldsymbol{x}}_1\right)$$

$$\tilde{\boldsymbol{P}}_F = \tilde{\boldsymbol{P}}_1 - \frac{\tilde{\boldsymbol{P}}_1 \boldsymbol{u}\boldsymbol{u}^\top \tilde{\boldsymbol{P}}_1}{\sigma_1^2 + \sigma_m^2}$$

$$\boldsymbol{K} = \frac{\tilde{\boldsymbol{P}}_1 \boldsymbol{u}}{\sigma_1^2 + \sigma_m^2}$$

$$\sigma_1^2 = \boldsymbol{u}^\top \tilde{\boldsymbol{P}}_1 \boldsymbol{u}$$

# Filtering of an autonomous measurement



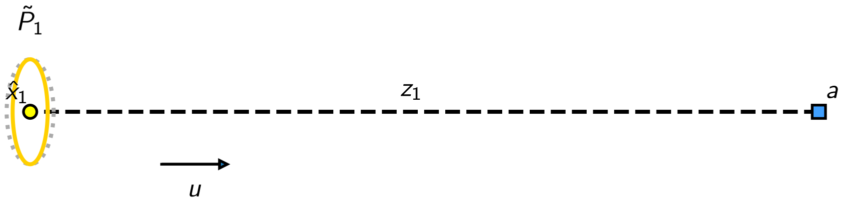

$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left(z_1 - \boldsymbol{u}^\top \hat{\boldsymbol{x}}_1\right)$$

$$\tilde{\boldsymbol{P}}_F = \tilde{\boldsymbol{P}}_1 - \frac{\tilde{\boldsymbol{P}}_1 \boldsymbol{u}\boldsymbol{u}^\top \tilde{\boldsymbol{P}}_1}{\sigma_1^2 + \sigma_m^2}$$

$$\boldsymbol{K} = \frac{\tilde{\boldsymbol{P}}_1 \boldsymbol{u}}{\sigma_1^2 + \sigma_m^2}$$

$$\sigma_1^2 = \boldsymbol{u}^\top \tilde{\boldsymbol{P}}_1 \boldsymbol{u}$$

# Filtering of an autonomous measurement



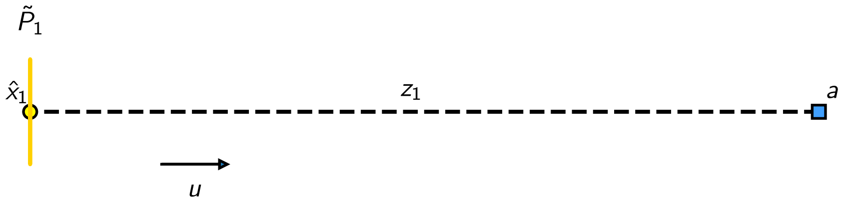

$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left(z_1 - \boldsymbol{u}^\top \hat{\boldsymbol{x}}_1\right)$$

$$\tilde{\boldsymbol{P}}_F = \tilde{\boldsymbol{P}}_1 - \frac{\tilde{\boldsymbol{P}}_1 \boldsymbol{u}\boldsymbol{u}^\top \tilde{\boldsymbol{P}}_1}{\sigma_1^2 + \sigma_m^2}$$

$$\boldsymbol{K} = \frac{\tilde{\boldsymbol{P}}_1 \boldsymbol{u}}{\sigma_1^2 + \sigma_m^2}$$

$$\sigma_1^2 = \boldsymbol{u}^\top \tilde{\boldsymbol{P}}_1 \boldsymbol{u}$$

# Filtering of an autonomous measurement



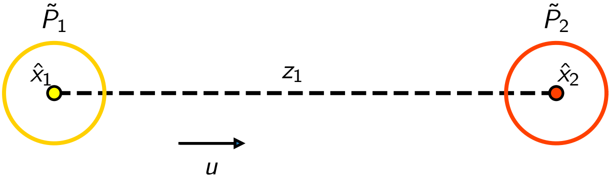

$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left(z_1 - \boldsymbol{u}^\top \hat{\boldsymbol{x}}_1\right)$$

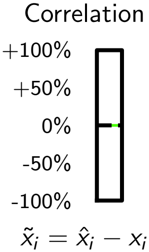
$$\tilde{\boldsymbol{P}}_F = \tilde{\boldsymbol{P}}_1 - \frac{\tilde{\boldsymbol{P}}_1 \boldsymbol{u}\boldsymbol{u}^\top \tilde{\boldsymbol{P}}_1}{\sigma_1^2 + \sigma_m^2}$$

$$\boldsymbol{K} = \frac{\tilde{\boldsymbol{P}}_1 \boldsymbol{u}}{\sigma_1^2 + \sigma_m^2}$$

$$\sigma_1^2 = \boldsymbol{u}^\top \tilde{\boldsymbol{P}}_1 \boldsymbol{u}$$

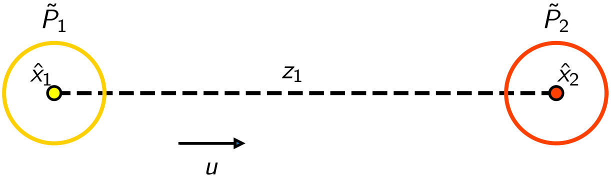
# Filtering of a cooperative measurement

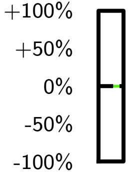



$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left[z_1 - \boldsymbol{u}^\intercal\left(\hat{\boldsymbol{x}}_1 - \hat{\boldsymbol{x}}_2\right)\right]$$

# Filtering of a cooperative measurement



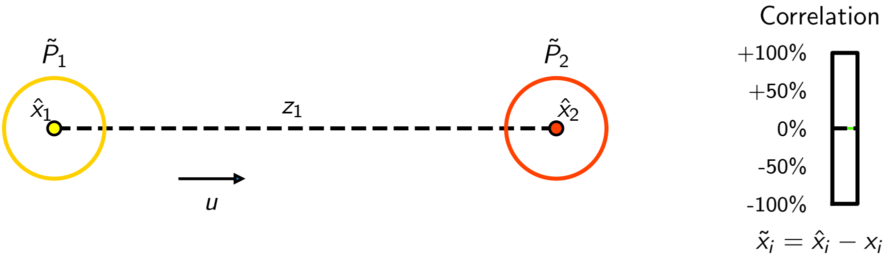

$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left[z_1 - \boldsymbol{u}^\top\left(\hat{\boldsymbol{x}}_1 - \hat{\boldsymbol{x}}_2\right)\right]$$

$$\boldsymbol{K} = \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

$$\sigma_2^2 = \boldsymbol{u}^\top \tilde{\boldsymbol{P}}_2 \boldsymbol{u}$$

$$\gamma_{12}^2 = \boldsymbol{u}^\top \tilde{\boldsymbol{P}}_{12} \boldsymbol{u}$$

# Filtering of a cooperative measurement




$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left[z_1 - \boldsymbol{u}^\mathsf{T}\left(\hat{\boldsymbol{x}}_1 - \hat{\boldsymbol{x}}_2\right)\right]$$

$$\tilde{\boldsymbol{P}}_F = \tilde{\boldsymbol{P}}_1 - \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}\boldsymbol{u}^\mathsf{T}\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)^\mathsf{T}}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

$$\boldsymbol{K} = \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

$$\sigma_2^2 = \boldsymbol{u}^\mathsf{T}\tilde{\boldsymbol{P}}_2\boldsymbol{u}$$

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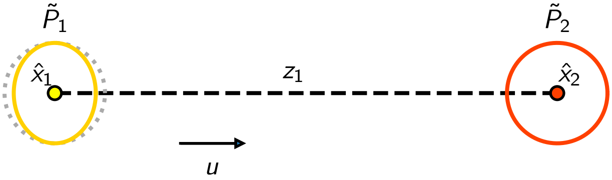
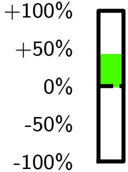
# Filtering of a cooperative measurement




$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left[z_1 - \boldsymbol{u}^\intercal\left(\hat{\boldsymbol{x}}_1 - \hat{\boldsymbol{x}}_2\right)\right]$$

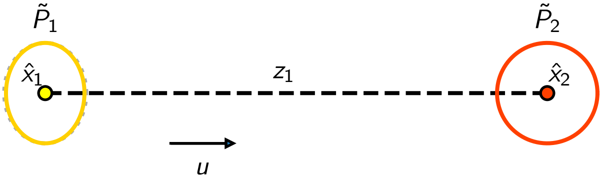
$$\tilde{\boldsymbol{P}}_F = \tilde{\boldsymbol{P}}_1 - \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}\boldsymbol{u}^\intercal\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)^\intercal}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

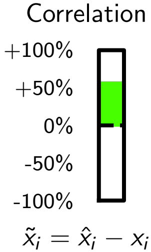
$$\boldsymbol{K} = \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

$$\sigma_2^2 = \boldsymbol{u}^\intercal \tilde{\boldsymbol{P}}_2 \boldsymbol{u}$$

$$\gamma_{12}^2 = \boldsymbol{u}^\intercal \tilde{\boldsymbol{P}}_{12} \boldsymbol{u}$$

# Filtering of a cooperative measurement




$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left[z_1 - \boldsymbol{u}^\mathsf{T}\left(\hat{\boldsymbol{x}}_1 - \hat{\boldsymbol{x}}_2\right)\right]$$

$$\tilde{\boldsymbol{P}}_F = \tilde{\boldsymbol{P}}_1 - \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}\boldsymbol{u}^\mathsf{T}\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)^\mathsf{T}}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

$$\boldsymbol{K} = \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

$$\sigma_2^2 = \boldsymbol{u}^\mathsf{T}\tilde{\boldsymbol{P}}_2\boldsymbol{u}$$

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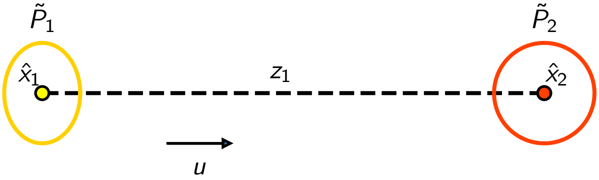
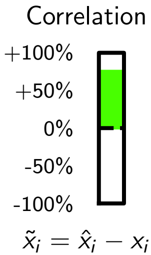
# Filtering of a cooperative measurement




$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left[z_1 - \boldsymbol{u}^\intercal\left(\hat{\boldsymbol{x}}_1 - \hat{\boldsymbol{x}}_2\right)\right]$$

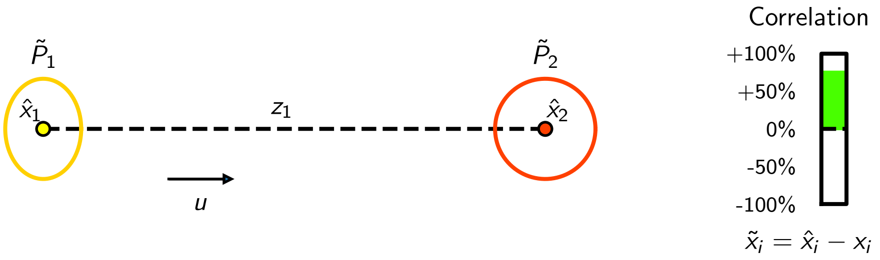
$$\tilde{\boldsymbol{P}}_F = \tilde{\boldsymbol{P}}_1 - \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}\boldsymbol{u}^\intercal\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)^\intercal}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

$$\boldsymbol{K} = \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

$$\sigma_2^2 = \boldsymbol{u}^\intercal\tilde{\boldsymbol{P}}_2\boldsymbol{u}$$

$$\gamma_{12}^2 = \boldsymbol{u}^\intercal\tilde{\boldsymbol{P}}_{12}\boldsymbol{u}$$

# Can we implement such a correction?

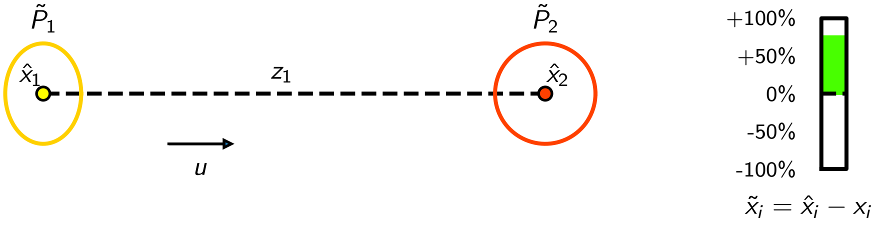



$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left[z_1 - \boldsymbol{u}^\mathsf{T}\left(\hat{\boldsymbol{x}}_1 - \hat{\boldsymbol{x}}_2\right)\right]$$

$$\boldsymbol{K} = \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

$$\tilde{\boldsymbol{P}}_F = \tilde{\boldsymbol{P}}_1 - \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}\boldsymbol{u}^\mathsf{T}\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)^\mathsf{T}}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

# Can we implement such a correction?



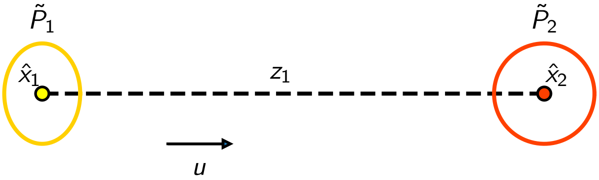

$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left[z_1 - \boldsymbol{u}^\mathsf{T}\left(\hat{\boldsymbol{x}}_1 - \hat{\boldsymbol{x}}_2\right)\right]$$

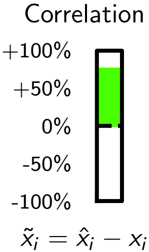
$$\boldsymbol{K} = \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

$(\hat{\boldsymbol{x}}_1, \tilde{\boldsymbol{P}}_1)$ known

$$\tilde{\boldsymbol{P}}_F = \tilde{\boldsymbol{P}}_1 - \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}\boldsymbol{u}^\mathsf{T}\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)^\mathsf{T}}{\sigma_2^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

# Can we implement such a correction?




$$\hat{\boldsymbol{x}}_F = \hat{\boldsymbol{x}}_1 + \boldsymbol{K}\left[z_1 - \boldsymbol{u}^\mathsf{T}\left(\hat{\boldsymbol{x}}_1 - \hat{\boldsymbol{x}}_2\right)\right]$$

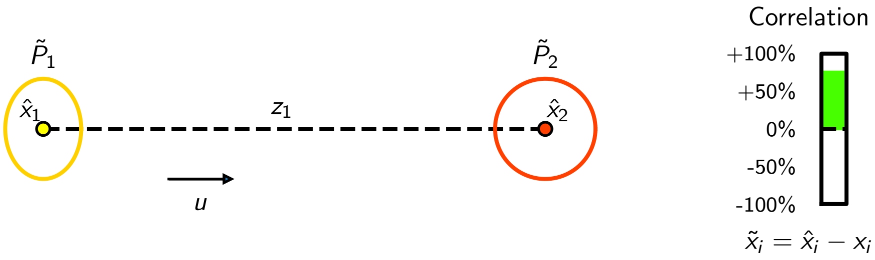
$$\boldsymbol{K} = \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

$$\tilde{\boldsymbol{P}}_F = \tilde{\boldsymbol{P}}_1 - \frac{\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)\boldsymbol{u}\boldsymbol{u}^\mathsf{T}\left(\tilde{\boldsymbol{P}}_1 - \tilde{\boldsymbol{P}}_{12}\right)^\mathsf{T}}{\sigma_2^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

$(\hat{\boldsymbol{x}}_1, \tilde{\boldsymbol{P}}_1)$ known

$(\hat{\boldsymbol{x}}_2, \tilde{\boldsymbol{P}}_2)$ known

# Can we implement such a correction?




$$\hat{\mathbf{x}}_F = \hat{\mathbf{x}}_1 + \mathbf{K}\left[z_1 - \mathbf{u}^\mathsf{T}\left(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2\right)\right]$$

$$\mathbf{K} = \frac{\left(\tilde{\mathbf{P}}_1 - \tilde{\mathbf{P}}_{12}\right)\mathbf{u}}{\sigma_1^2 + \sigma_2^2 - 2\gamma_{12} + \sigma_m^2}$$

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| | |
|---|---|
| $(\hat{\mathbf{x}}_1, \tilde{\mathbf{P}}_1)$ | known |
| $(\hat{\mathbf{x}}_2, \tilde{\mathbf{P}}_2)$ | known |
| $\tilde{\mathbf{P}}_{12}$ | **unknown!** |

# Conservative fusion (1/2)

$(\hat{\boldsymbol{x}}_1, \tilde{\boldsymbol{P}}_1), \ldots, (\hat{\boldsymbol{x}}_N, \tilde{\boldsymbol{P}}_N)$ known.

# Conservative fusion (1/2)

$(\hat{\boldsymbol{x}}_1, \tilde{\boldsymbol{P}}_1), \ldots, (\hat{\boldsymbol{x}}_N, \tilde{\boldsymbol{P}}_N)$ known.

$$\tilde{\boldsymbol{P}}_{\mathrm{c}} = \begin{bmatrix} \tilde{\boldsymbol{P}}_1 & \tilde{\boldsymbol{P}}_{12} & \cdots & \tilde{\boldsymbol{P}}_{1N} \\ * & \tilde{\boldsymbol{P}}_2 & \cdots & \tilde{\boldsymbol{P}}_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & * & \tilde{\boldsymbol{P}}_N \end{bmatrix} \text{ unknown!}$$

$$\tilde{\boldsymbol{P}}_{\mathrm{c}} \in \mathcal{A} \subseteq \left\{ \boldsymbol{P}_{\mathrm{c}} \succeq \boldsymbol{0} \mid \forall i,\ \boldsymbol{P}_i = \tilde{\boldsymbol{P}}_i \right\}$$

# Conservative fusion (1/2)

$(\hat{\boldsymbol{x}}_1, \tilde{\boldsymbol{P}}_1), \ldots, (\hat{\boldsymbol{x}}_N, \tilde{\boldsymbol{P}}_N)$ known.

$\hat{\boldsymbol{x}}_F(\boldsymbol{K}) = \sum_i \boldsymbol{K}_i \hat{\boldsymbol{x}}_i = \boldsymbol{K}\hat{\boldsymbol{x}}_\mathrm{c}$ with $\sum_i \boldsymbol{K}_i = \boldsymbol{I}$

$\tilde{\boldsymbol{P}}_F = \boldsymbol{K}\tilde{\boldsymbol{P}}_\mathrm{c}\boldsymbol{K}^\intercal = \tilde{\boldsymbol{P}}_F(\boldsymbol{K}, \tilde{\boldsymbol{P}}_\mathrm{c})$

$$\tilde{\boldsymbol{P}}_\mathrm{c} = \begin{bmatrix} \tilde{\boldsymbol{P}}_1 & \tilde{\boldsymbol{P}}_{12} & \cdots & \tilde{\boldsymbol{P}}_{1N} \\ * & \tilde{\boldsymbol{P}}_2 & \cdots & \tilde{\boldsymbol{P}}_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & * & \tilde{\boldsymbol{P}}_N \end{bmatrix} \text{ unknown!}$$

$$\tilde{\boldsymbol{P}}_\mathrm{c} \in \mathcal{A} \subseteq \left\{ \boldsymbol{P}_\mathrm{c} \succeq \boldsymbol{0} \mid \forall i,\ \boldsymbol{P}_i = \tilde{\boldsymbol{P}}_i \right\}$$

# Conservative fusion (1/2)

$(\hat{\boldsymbol{x}}_1, \tilde{\boldsymbol{P}}_1), \ldots, (\hat{\boldsymbol{x}}_N, \tilde{\boldsymbol{P}}_N)$ known.

$\hat{\boldsymbol{x}}_F(\boldsymbol{K}) = \sum_i \boldsymbol{K}_i \hat{\boldsymbol{x}}_i = \boldsymbol{K}\hat{\boldsymbol{x}}_\mathrm{c}$ with $\sum_i \boldsymbol{K}_i = \boldsymbol{I}$

$\tilde{\boldsymbol{P}}_F = \boldsymbol{K}\tilde{\boldsymbol{P}}_\mathrm{c}\boldsymbol{K}^\intercal = \tilde{\boldsymbol{P}}_F(\boldsymbol{K}, \tilde{\boldsymbol{P}}_\mathrm{c})$

$(\hat{\boldsymbol{x}}_F(\boldsymbol{K}), \boldsymbol{B}_F)$ conservative fused estimate
$\forall \boldsymbol{P}_\mathrm{c} \in \mathcal{A},\ \tilde{\boldsymbol{P}}_F(\boldsymbol{K}, \boldsymbol{P}_\mathrm{c}) \preceq \boldsymbol{B}_F$

$$\tilde{\boldsymbol{P}}_\mathrm{c} = \begin{bmatrix} \tilde{\boldsymbol{P}}_1 & \tilde{\boldsymbol{P}}_{12} & \cdots & \tilde{\boldsymbol{P}}_{1N} \\ * & \tilde{\boldsymbol{P}}_2 & \cdots & \tilde{\boldsymbol{P}}_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & * & \tilde{\boldsymbol{P}}_N \end{bmatrix} \text{ unknown!}$$

$$\tilde{\boldsymbol{P}}_\mathrm{c} \in \mathcal{A} \subseteq \left\{ \boldsymbol{P}_\mathrm{c} \succeq \boldsymbol{0} \mid \forall i,\ \boldsymbol{P}_i = \tilde{\boldsymbol{P}}_i \right\}$$

# Conservative fusion (1/2)

$(\hat{\boldsymbol{x}}_1, \tilde{\boldsymbol{P}}_1), \dots, (\hat{\boldsymbol{x}}_N, \tilde{\boldsymbol{P}}_N)$ known.

$\hat{\boldsymbol{x}}_F(\boldsymbol{K}) = \sum_i \boldsymbol{K}_i \hat{\boldsymbol{x}}_i = \boldsymbol{K}\hat{\boldsymbol{x}}_c$ with $\sum_i \boldsymbol{K}_i = \boldsymbol{I}$

$\tilde{\boldsymbol{P}}_F = \boldsymbol{K}\tilde{\boldsymbol{P}}_c\boldsymbol{K}^\intercal = \tilde{\boldsymbol{P}}_F(\boldsymbol{K}, \tilde{\boldsymbol{P}}_c)$

$$\tilde{\boldsymbol{P}}_c = \begin{bmatrix} \tilde{\boldsymbol{P}}_1 & \tilde{\boldsymbol{P}}_{12} & \cdots & \tilde{\boldsymbol{P}}_{1N} \\ * & \tilde{\boldsymbol{P}}_2 & \cdots & \tilde{\boldsymbol{P}}_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & * & \tilde{\boldsymbol{P}}_N \end{bmatrix} \text{ unknown!}$$

$(\hat{\boldsymbol{x}}_F(\boldsymbol{K}), \boldsymbol{B}_F)$ conservative fused estimate
$\forall \boldsymbol{P}_c \in \mathcal{A},\ \tilde{\boldsymbol{P}}_F(\boldsymbol{K}, \boldsymbol{P}_c) \preceq \boldsymbol{B}_F$

$$\tilde{\boldsymbol{P}}_c \in \mathcal{A} \subseteq \left\{ \boldsymbol{P}_c \succeq \boldsymbol{0} \mid \forall i,\ \boldsymbol{P}_i = \tilde{\boldsymbol{P}}_i \right\}$$

$$\begin{cases} \arg\min_{\boldsymbol{K},\boldsymbol{B}} & J(\boldsymbol{B}) \\ \text{subject to:} & \forall \boldsymbol{P}_c \in \mathcal{A},\ \boldsymbol{K}\boldsymbol{P}_c\boldsymbol{K}^\intercal \preceq \boldsymbol{B} \end{cases}$$

---

Robin Forsling et al. "Conservative linear unbiased estimation under partially known covariances". In: *IEEE Transactions on Signal Processing* 70 (2022), pp. 3123–3135

# Conservative fusion (2/2)

$$\begin{cases} \text{Find} & \boldsymbol{K}, \boldsymbol{B} \\ \text{subject to:} & \forall \boldsymbol{P}_{\mathrm{c}} \in \mathcal{A},\ \boldsymbol{K}\boldsymbol{P}_{\mathrm{c}}\boldsymbol{K}^{\intercal} \preceq \boldsymbol{B} \end{cases}$$

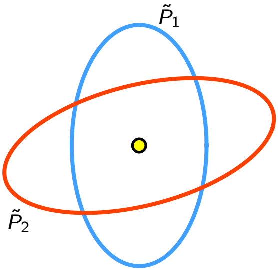
1. Find $\boldsymbol{B}_{\mathrm{c}}$ such that $\forall \boldsymbol{P}_{\mathrm{c}} \in \mathcal{A}$, $\boldsymbol{P}_{\mathrm{c}} \preceq \boldsymbol{B}_{\mathrm{c}}$.
2. Optimize $\boldsymbol{K}$ for $\boldsymbol{B}_{\mathrm{c}}$.
3. Set $\boldsymbol{B}_F = \boldsymbol{K}\boldsymbol{B}_{\mathrm{c}}\boldsymbol{K}^{\intercal}$.

$$\tilde{\boldsymbol{P}}_F = \boldsymbol{K}\tilde{\boldsymbol{P}}_{\mathrm{c}}\boldsymbol{K}^{\intercal} \preceq \boldsymbol{B}_F.$$

# Covariance Intersection

$$\begin{bmatrix} \tilde{\boldsymbol{P}}_1 & \tilde{\boldsymbol{P}}_{12} & \cdots & \tilde{\boldsymbol{P}}_{1N} \\ \tilde{\boldsymbol{P}}_{21} & \tilde{\boldsymbol{P}}_2 & \cdots & \tilde{\boldsymbol{P}}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\boldsymbol{P}}_{N1} & \tilde{\boldsymbol{P}}_{N2} & \cdots & \tilde{\boldsymbol{P}}_N \end{bmatrix} \preceq \begin{bmatrix} \frac{1}{\omega_1}\tilde{\boldsymbol{P}}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \frac{1}{\omega_2}\tilde{\boldsymbol{P}}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \frac{1}{\omega_N}\tilde{\boldsymbol{P}}_N \end{bmatrix}$$

with $\omega_i \geq 0$ and $\sum_i \omega_i = 1$.




$$\boldsymbol{B}_F^{-1} = \sum \omega_i \tilde{\boldsymbol{P}}_i^{-1}$$

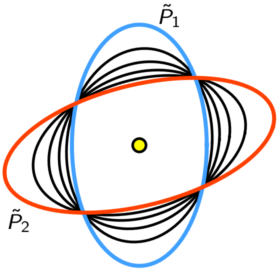
$$\boldsymbol{B}_F^{-1}\hat{\boldsymbol{x}}_F = \sum \omega_i \tilde{\boldsymbol{P}}_i^{-1}\hat{\boldsymbol{x}}_i$$

---

Simon J Julier and Jeffrey K Uhlmann. "A non-divergent estimation algorithm in the presence of unknown correlations". In: *Proceedings of the 1997 American Control Conference*. Vol. 4. IEEE. 1997, pp. 2369–2373

# Covariance Intersection

$$\begin{bmatrix} \tilde{\boldsymbol{P}}_1 & \tilde{\boldsymbol{P}}_{12} & \cdots & \tilde{\boldsymbol{P}}_{1N} \\ \tilde{\boldsymbol{P}}_{21} & \tilde{\boldsymbol{P}}_2 & \cdots & \tilde{\boldsymbol{P}}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\boldsymbol{P}}_{N1} & \tilde{\boldsymbol{P}}_{N2} & \cdots & \tilde{\boldsymbol{P}}_N \end{bmatrix} \preceq \begin{bmatrix} \frac{1}{\omega_1}\tilde{\boldsymbol{P}}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \frac{1}{\omega_2}\tilde{\boldsymbol{P}}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \frac{1}{\omega_N}\tilde{\boldsymbol{P}}_N \end{bmatrix}$$

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$$\boldsymbol{B}_F^{-1} = \sum \omega_i \tilde{\boldsymbol{P}}_i^{-1}$$

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# Exploiting the known components

$$\hat{\boldsymbol{x}}_i = (\boldsymbol{I} - \boldsymbol{K}_i\boldsymbol{H})\,\boldsymbol{F}\hat{\boldsymbol{x}}_i^- + \boldsymbol{K}_i\boldsymbol{z}_i,$$
$$\tilde{\boldsymbol{x}}_i = (\boldsymbol{I} - \boldsymbol{K}_i\boldsymbol{H})\,(\boldsymbol{F}\tilde{\boldsymbol{x}}_i^- - \boldsymbol{w}_i) + \boldsymbol{K}_i\boldsymbol{v}_i,$$

$\tilde{\boldsymbol{x}}_i^-$ : previous error,
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$$\tilde{\boldsymbol{x}}_i = \tilde{\boldsymbol{x}}_i^{(1)} + \tilde{\boldsymbol{x}}_i^{(2)}$$

$(\tilde{\boldsymbol{x}}_i^{(1)}, \tilde{\boldsymbol{P}}_i^{(1)})$ with unknown correlations

$(\tilde{\boldsymbol{x}}_i^{(2)}, \tilde{\boldsymbol{P}}_i^{(2)})$ with known second order moments

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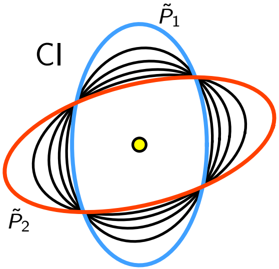
$$\tilde{\boldsymbol{P}}_{\mathrm{c}} = \begin{bmatrix} \tilde{\boldsymbol{P}}_1^{(1)} & \tilde{\boldsymbol{P}}_{12}^{(1)} & \cdots & \tilde{\boldsymbol{P}}_{1N}^{(1)} \\ * & \tilde{\boldsymbol{P}}_2^{(1)} & \cdots & \tilde{\boldsymbol{P}}_{2N}^{(1)} \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & * & \tilde{\boldsymbol{P}}_N^{(1)} \end{bmatrix} + \tilde{\boldsymbol{P}}_{\mathrm{c}}^{(2)} + \tilde{\boldsymbol{P}}_{\mathrm{c}}^{(12)} + \tilde{\boldsymbol{P}}_{\mathrm{c}}^{(21)}$$

# Extended Split Covariance Intersection

$$\hat{\boldsymbol{x}}_i = (\boldsymbol{I} - \boldsymbol{K}_i\boldsymbol{H})\,\boldsymbol{F}\hat{\boldsymbol{x}}_i^- + \boldsymbol{K}_i\boldsymbol{z}_i, \qquad \tilde{\boldsymbol{x}}_i = (\boldsymbol{I} - \boldsymbol{K}_i\boldsymbol{H})\,(\boldsymbol{F}\tilde{\boldsymbol{x}}_i^- - \boldsymbol{w}_i) + \boldsymbol{K}_i\boldsymbol{v}_i$$




Without any structure

# Extended Split Covariance Intersection

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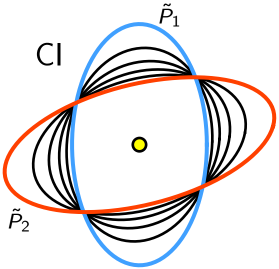
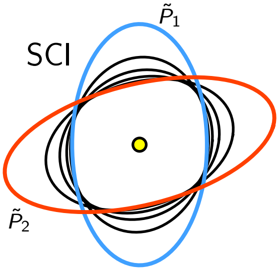



Without any structure




By considering the independent components


Simon J Julier and Jeffrey K Uhlmann. "Simultaneous localisation and map building using split covariance intersection". In: *Proceedings 2001 IEEE/RSJ International Conference on Intelligent Robots and Systems.* Vol. 3. 2001, pp. 1257–1262

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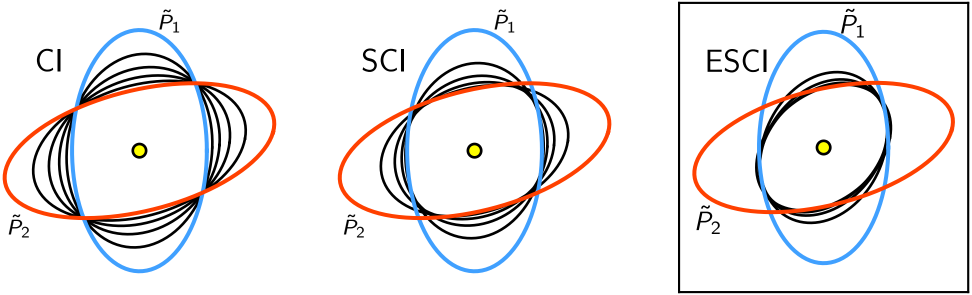



Without any structure

By considering the independent components

By considering the indep. and common components

Simon J Julier and Jeffrey K Uhlmann. "Simultaneous localisation and map building using split covariance intersection". In: *Proceedings 2001 IEEE/RSJ International Conference on Intelligent Robots and Systems*. Vol. 3. 2001, pp. 1257–1262

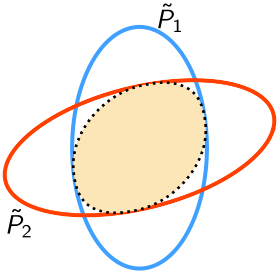
# Optimality for two estimators

$$\begin{cases} \arg\min_{\boldsymbol{K},\boldsymbol{B}} & J(\boldsymbol{B}) \\ \text{subject to:} & \forall \boldsymbol{P}_{\mathrm{c}} \in \mathcal{A},\ \boldsymbol{K}\boldsymbol{P}_{\mathrm{c}}\boldsymbol{K}^{\intercal} \preceq \boldsymbol{B} \end{cases} \qquad (\mathcal{P})$$

$$\mathcal{A} = \left\{ \boldsymbol{P}_{\mathrm{c}}^{(1)} + \tilde{\boldsymbol{P}}_{\mathrm{c}}^{(2)} + \tilde{\boldsymbol{P}}_{\mathrm{c}}^{(12)} + \tilde{\boldsymbol{P}}_{\mathrm{c}}^{(21)} \mid \boldsymbol{P}_{\mathrm{c}}^{(1)} \succeq \boldsymbol{0},\ \boldsymbol{P}_i^{(1)} = \tilde{\boldsymbol{P}}_i^{(1)} \right\}$$

**Theorem [C2]:**

For the fusion of 2 estimators, the ESCI fusion provides a solution to $(\mathcal{P})$.




[C2] Colin Cros et al. "Revisiting Split Covariance Intersection: Correlated Components and Optimality". In: *IEEE Transactions on Automatic Control* (2025), pp. 1–16

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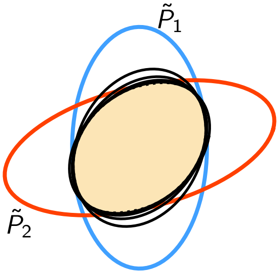


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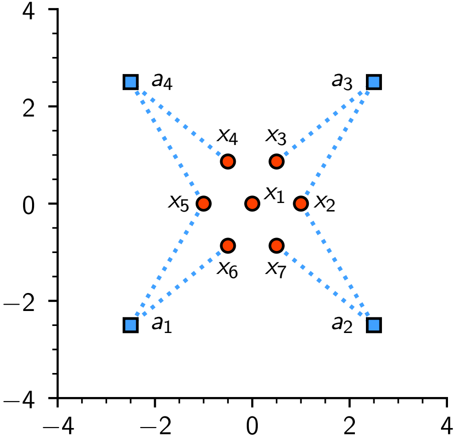
# Application



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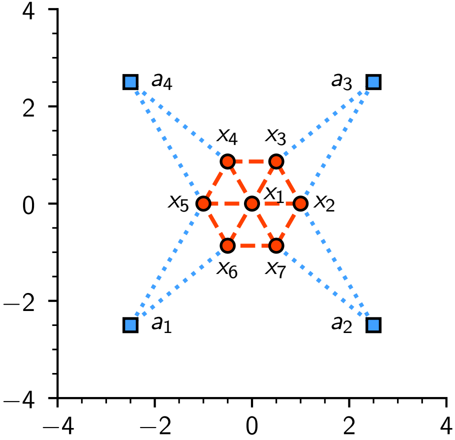

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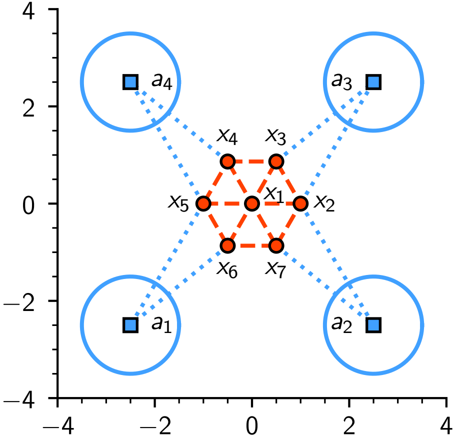
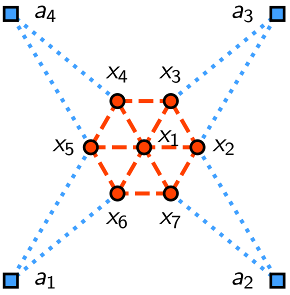
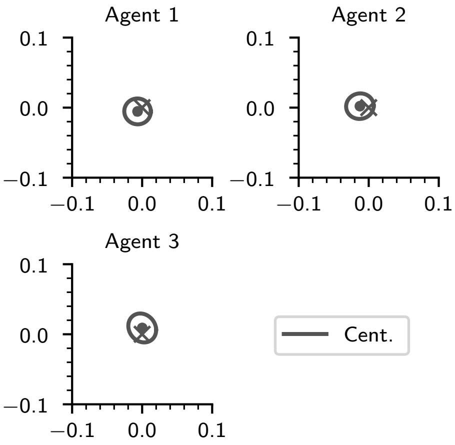


# Results: Qualitative



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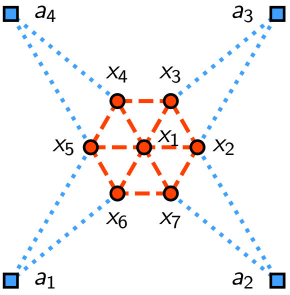


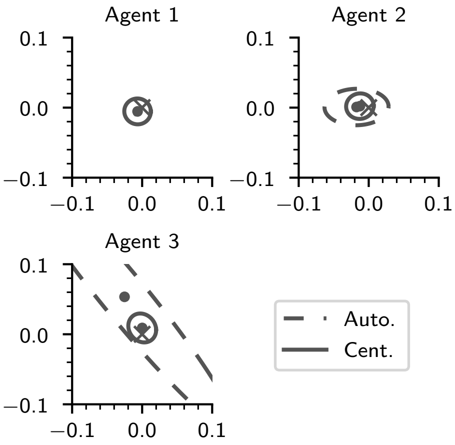

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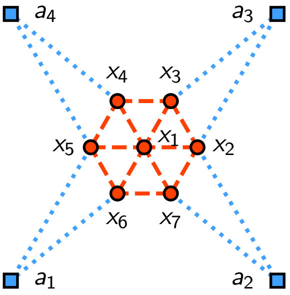


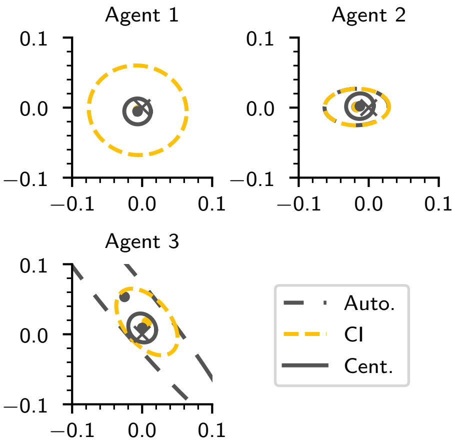

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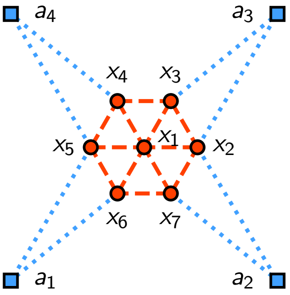


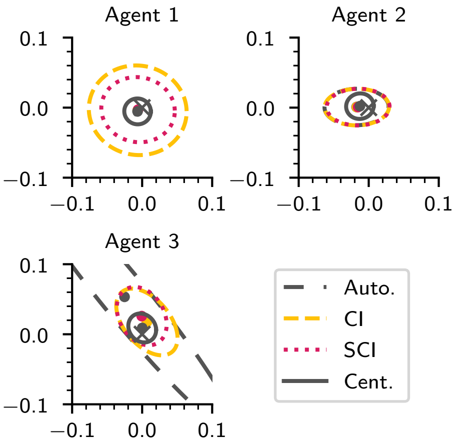

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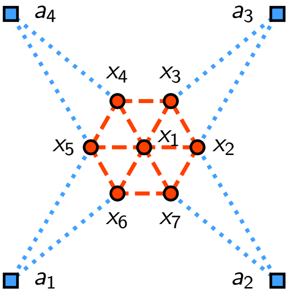


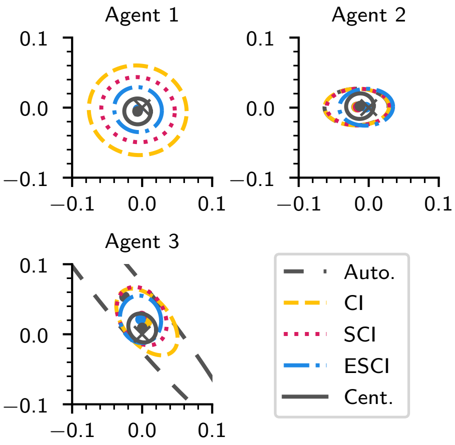

# Take home message

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2. We have extended the Split Covariance Intersection fusion in order to exploit the common components of the errors. The new fusion provides tighter bounds.

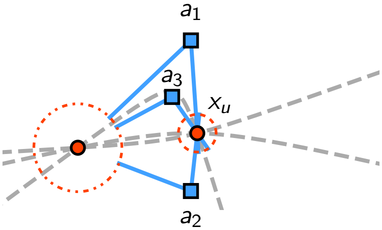
3. We have proved that this fusion provides the optimal conservative bounds for the fusion of two estimators.

# Table of contents

# Uniqueness of the solution

**Result:**

$\Gamma$ Pseudorange Rigid $\Leftrightarrow \exists G_D \cup G_S = \Gamma$ with $G_D$ Distance Rigid and $G_S$ connected.



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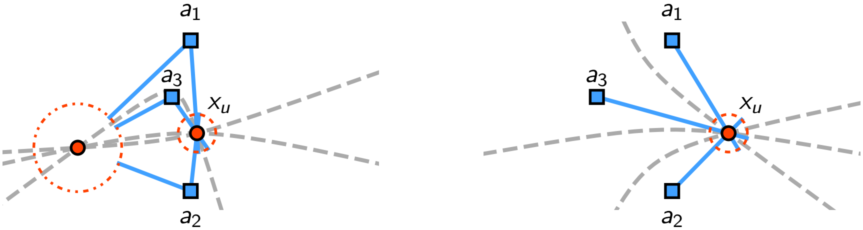

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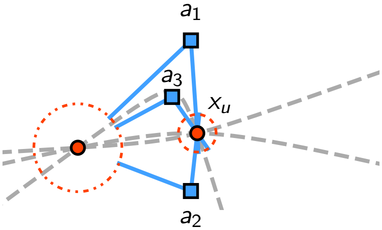
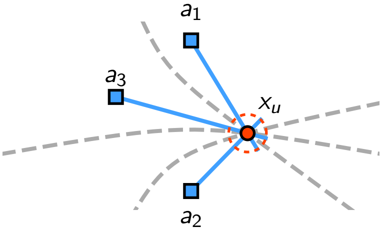



**Conjecture:**

$\Gamma$ Pseudorange Globally Rigid $\Leftarrow \exists G_D \cup G_S = \Gamma$ with $G_D$ Distance Globally Rigid and $G_S$ connected.

# Filtering improvement: Diffusion Kalman Filtering

$$\tilde{\boldsymbol{x}}_i = \tilde{\boldsymbol{x}}_i^{(1)} + \tilde{\boldsymbol{x}}_i^{(2)}$$

**Parameters sent:**

Current algorithm:

- Estimator $\hat{\boldsymbol{x}}_i$;
- Covariance $\tilde{\boldsymbol{P}}_i$;
- Covariance of the 2nd comp. $\tilde{\boldsymbol{P}}_i^{(2)}$.

Diffusion Kalman Filtering algorithm:

- Estimator $\hat{\boldsymbol{x}}_i$;
- Covariance $\tilde{\boldsymbol{P}}_i$;
- Covariance of the 2nd comp. $\tilde{\boldsymbol{P}}_i^{(2)}$;
- Information vector of the 2nd comp. $(\boldsymbol{P}_i^{(2)})^{-1}\hat{\boldsymbol{x}}_i^{(2)}$.

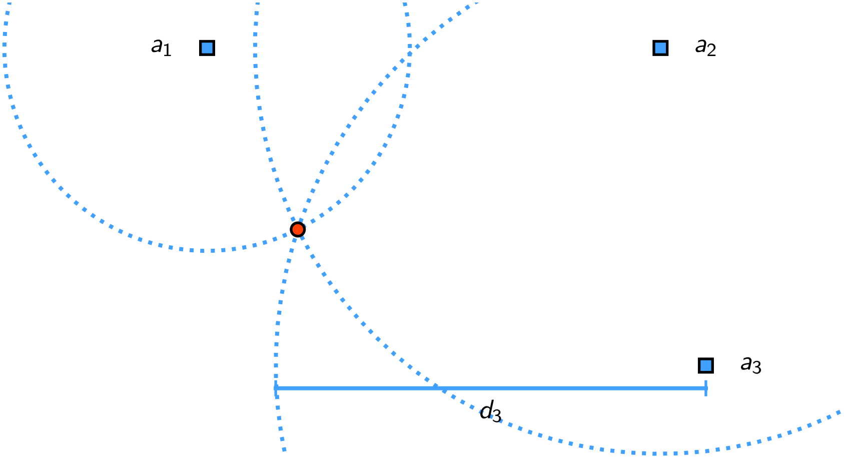
Jinwen Hu, Lihua Xie, and Cishen Zhang. "Diffusion Kalman filtering based on covariance intersection". In: *IEEE Transactions on Signal Processing* 60.2 (2011), pp. 891–902

# List of contributions


- Colin Cros et al. "Résolubilité du positionnement GNSS multi-agents". In: *GRETSI 2022-XXVIIIème Colloque GRETSI*. 2022
- Colin Cros et al. "Pseudorange Rigidity and Solvability of Cooperative GNSS Positioning". In: *IEEE Transactions on Control of Network Systems* (2024), pp. 1–12
- Colin Cros et al. "Revisiting Split Covariance Intersection: Correlated Components and Optimality". In: *IEEE Transactions on Automatic Control* (2025), pp. 1–16
- Colin Cros et al. "Intégration de mesures de distance entre agents de corrélation inconnue: cas unidimensionnel". In: *XXIXème Colloque GRETSI*. 2023
- Colin Cros et al. "Fusion of distance measurements between agents with unknown correlations". In: *IEEE Control Systems Letters* (2023)
- Colin Cros et al. "Split Covariance Intersection with Correlated Components for Distributed Estimation". In: *2024 27th International Conference on Information Fusion (FUSION)*. 2024, pp. 1–6

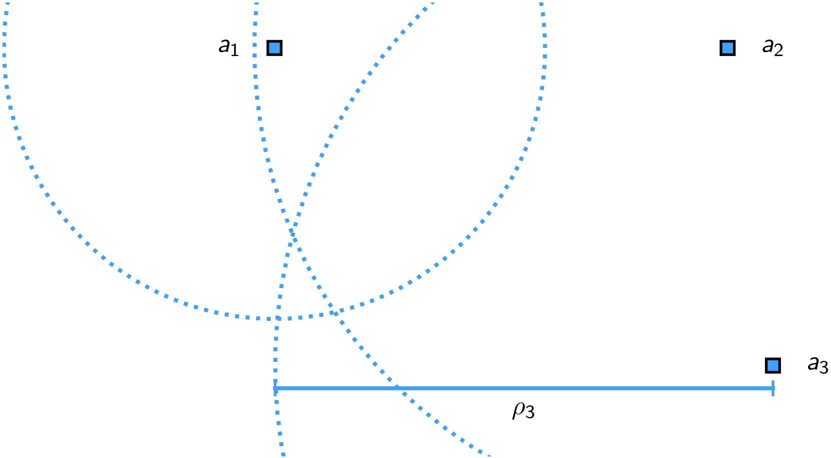
# Range-based positioning (Trilateration)

$$d_i = \|a_i - x_u\|$$



# Pseudorange-based positioning

$$\rho_i = \|a_i - x_u\| + \beta_u - \beta_a$$



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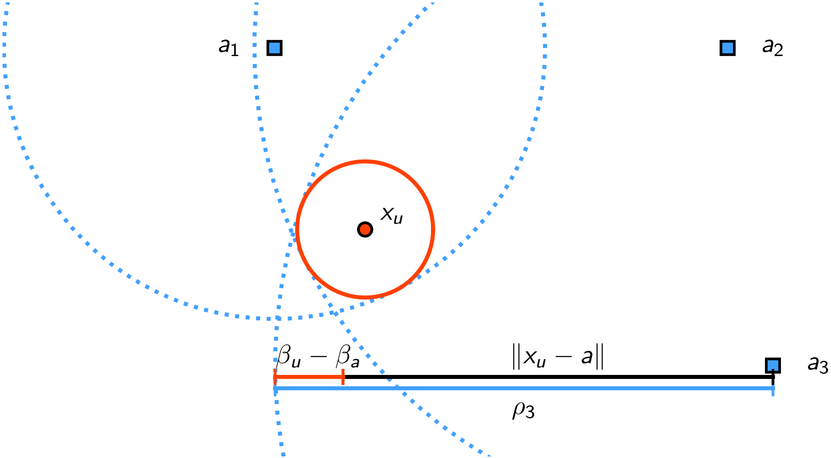
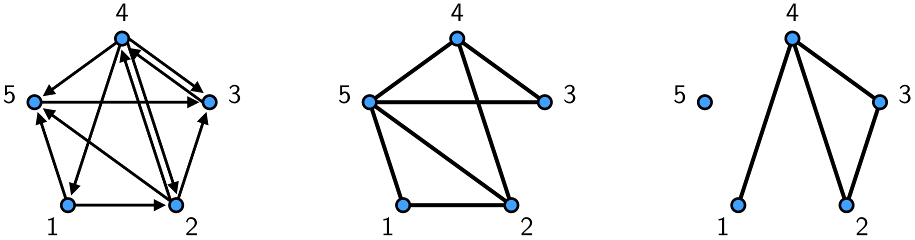


# Decompositions of pseudorange graphs




$\Gamma$ $\qquad$ $G_D$ $\qquad$ $G_S$

**Theorem:**

$$\operatorname{rank} \boldsymbol{R}_P(\Gamma, \boldsymbol{p}) = \max_{G_D \cup G_S = \tilde{\Gamma}} \operatorname{rank} \boldsymbol{R}_D(G_D, \boldsymbol{p}) + \operatorname{rank} \boldsymbol{R}_S(G_S, \boldsymbol{p})$$

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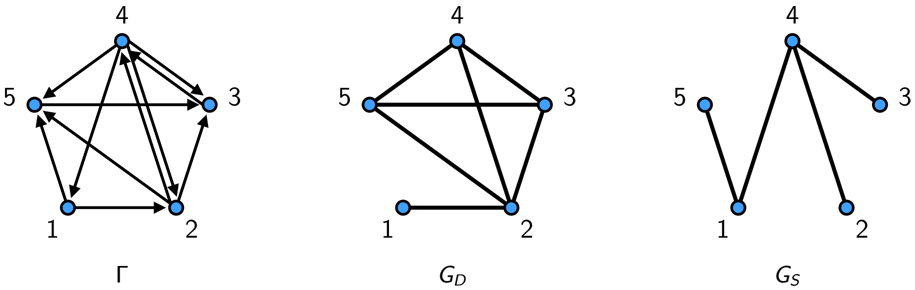


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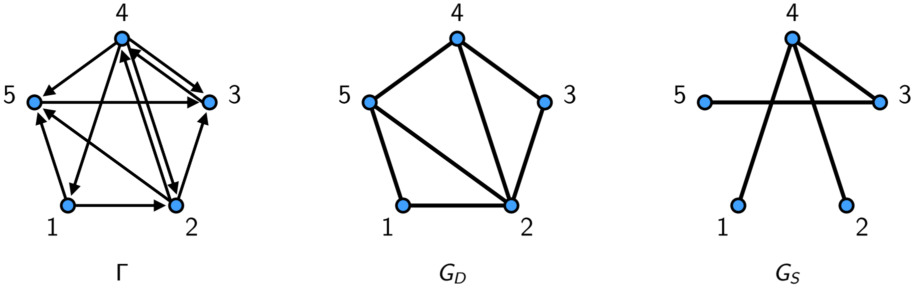


**Theorem:**

$$\operatorname{rank} \boldsymbol{R}_P(\Gamma, \boldsymbol{p}) = \max_{G_D \cup G_S = \tilde{\Gamma}} \operatorname{rank} \boldsymbol{R}_D(G_D, \boldsymbol{p}) + \operatorname{rank} \boldsymbol{R}_S(G_S, \boldsymbol{p})$$

# Split Covariance Intersection (1/2)

Autonomous correction:

$$\hat{\boldsymbol{x}}_i = (\boldsymbol{I} - \boldsymbol{K}_i \boldsymbol{H})\, \hat{\boldsymbol{x}}_i^{(pred)} + \boldsymbol{K}_i \boldsymbol{z}_i, \qquad \tilde{\boldsymbol{x}}_i = (\boldsymbol{I} - \boldsymbol{K}_i \boldsymbol{H})\, \tilde{\boldsymbol{x}}_i^{(pred)} + \boldsymbol{K}_i \boldsymbol{v}_i$$

$$\tilde{\boldsymbol{x}}_i = \tilde{\boldsymbol{x}}_i^{(1)} + \tilde{\boldsymbol{x}}_i^{(2)}$$

$(\tilde{\boldsymbol{x}}_i^{(1)}, \tilde{\boldsymbol{P}}_i^{(1)})$ with unknown correlations
$(\tilde{\boldsymbol{x}}_i^{(2)}, \tilde{\boldsymbol{P}}_i^{(2)})$ uncorrelated (independent)

$$\tilde{\boldsymbol{P}}_{\mathrm{c}} = \begin{bmatrix} \tilde{\boldsymbol{P}}_1^{(1)} + \tilde{\boldsymbol{P}}_1^{(2)} & \tilde{\boldsymbol{P}}_{12}^{(1)} & \cdots & \tilde{\boldsymbol{P}}_{1N}^{(1)} \\ \tilde{\boldsymbol{P}}_{21}^{(1)} & \tilde{\boldsymbol{P}}_2^{(1)} + \tilde{\boldsymbol{P}}_2^{(2)} & \cdots & \tilde{\boldsymbol{P}}_{2N}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\boldsymbol{P}}_{N1}^{(1)} & \tilde{\boldsymbol{P}}_{N2}^{(1)} & \cdots & \tilde{\boldsymbol{P}}_N^{(1)} + \tilde{\boldsymbol{P}}_N^{(2)} \end{bmatrix}$$

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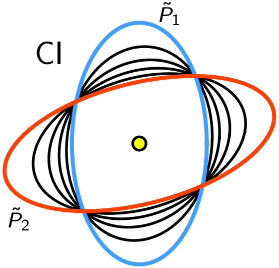
Simon J Julier and Jeffrey K Uhlmann. "Simultaneous localisation and map building using split covariance intersection". In: *Proceedings 2001 IEEE/RSJ International Conference on Intelligent Robots and Systems.* Vol. 3. 2001, pp. 1257–1262

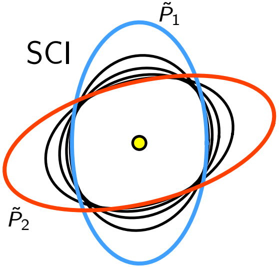
# Split Covariance Intersection (2/2)

$$\tilde{\boldsymbol{P}}_{\mathrm{c}} \preceq \begin{bmatrix} \frac{1}{\omega_1}\tilde{\boldsymbol{P}}_1^{(1)} + \tilde{\boldsymbol{P}}_1^{(2)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{\omega_N}\tilde{\boldsymbol{P}}_N^{(1)} + \tilde{\boldsymbol{P}}_N^{(2)} \end{bmatrix}$$

with $\omega_i \geq 0$ and $\sum_i \omega_i = 1$.

$$\boldsymbol{B}_F^{-1} = \sum \omega_i (\tilde{\boldsymbol{P}}_i^{(1)} + \omega_i \tilde{\boldsymbol{P}}_i^{(2)})^{-1}$$
$$\boldsymbol{B}_F^{-1}\hat{\boldsymbol{x}}_F = \sum \omega_i (\tilde{\boldsymbol{P}}_i^{(1)} + \omega_i \tilde{\boldsymbol{P}}_i^{(2)})^{-1}\hat{\boldsymbol{x}}_i$$



# Extended Split Covariance Intersection

$$\boldsymbol{B}_{\mathrm{c}}(\boldsymbol{\omega}) = \boldsymbol{B}_{\mathrm{c}}^{(1)}(\boldsymbol{\omega}) + \tilde{\boldsymbol{P}}_{\mathrm{c}}^{(2)} + \tilde{\boldsymbol{P}}_{\mathrm{c}}^{(12)} + \tilde{\boldsymbol{P}}_{\mathrm{c}}^{(21)}$$

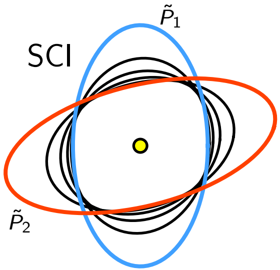
$$\boldsymbol{B}_F^{-1} = \boldsymbol{H}^\intercal \boldsymbol{B}_{\mathrm{c}}(\omega)^{-1} \boldsymbol{H}, \qquad \boldsymbol{H} = \mathbf{1}_N \otimes \boldsymbol{I},$$

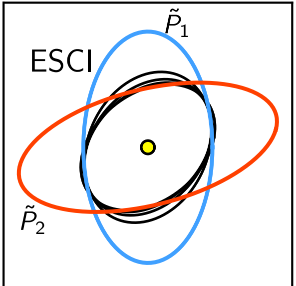
$$\boldsymbol{B}_F^{-1} \hat{\boldsymbol{x}}_F = \boldsymbol{H}^\intercal \boldsymbol{B}_{\mathrm{c}}(\omega)^{-1} \hat{\boldsymbol{x}}_{\mathrm{c}}$$

$$\tilde{\boldsymbol{x}}_i = \tilde{\boldsymbol{x}}_i^{(1)} + \tilde{\boldsymbol{x}}_i^{(ind)} + \boldsymbol{M}_i w,$$

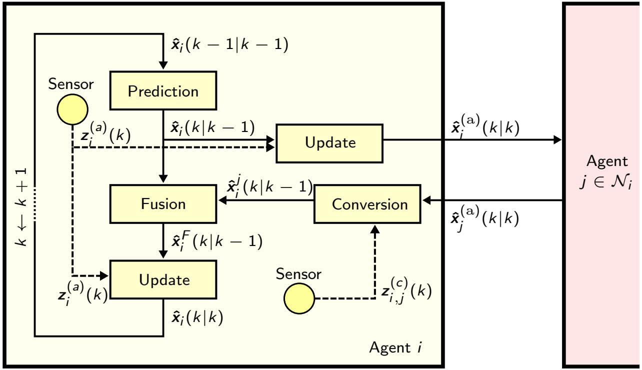
$$\tilde{\boldsymbol{P}}_{\mathrm{c}} = \tilde{\boldsymbol{P}}_{\mathrm{c}}^{(1)} + \tilde{\boldsymbol{P}}_{\mathrm{c}}^{(ind)} + \boldsymbol{M}_{\mathrm{c}} \boldsymbol{Q} \boldsymbol{M}_{\mathrm{c}}^\intercal.$$

Same complexity as for the SCI fusion.

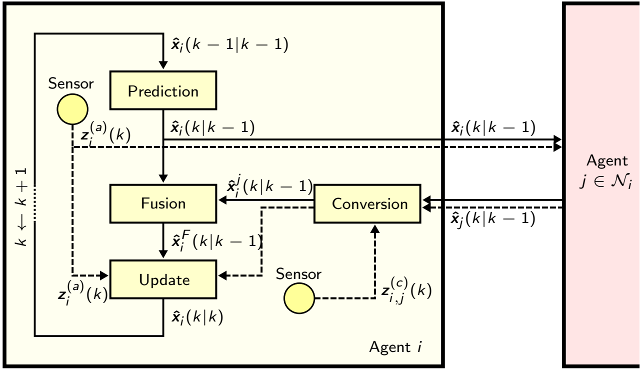


# Estimation algorithm

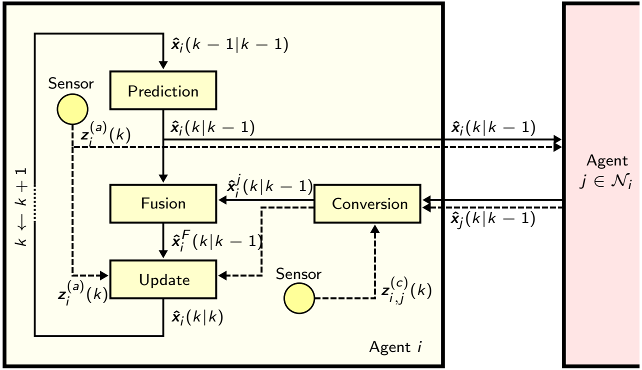


# Filtering improvement



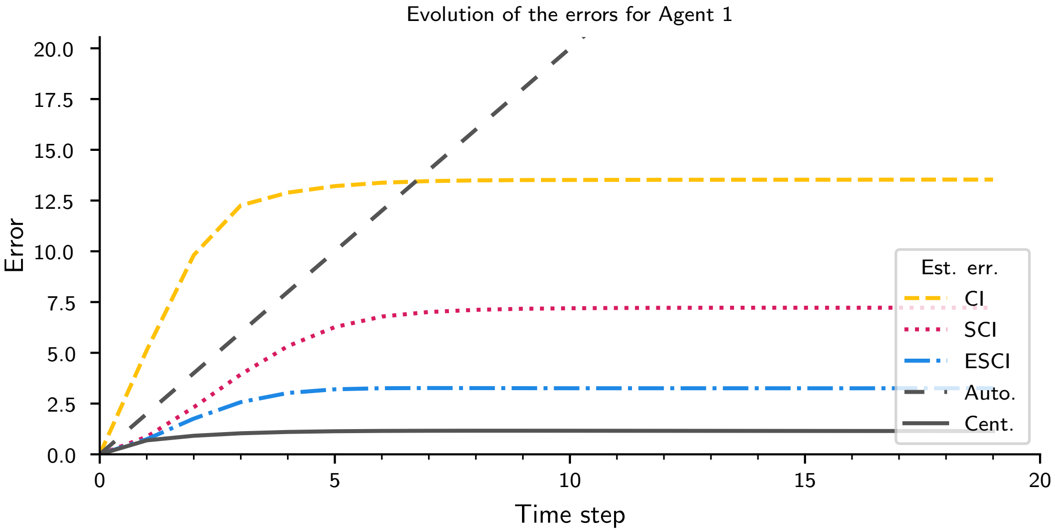
# Filtering improvement

Diffusion Kalman Filtering




Jinwen Hu, Lihua Xie, and Cishen Zhang. "Diffusion Kalman filtering based on covariance intersection". In: *IEEE Transactions on Signal Processing* 60.2 (2011), pp. 891–902

# Results: Quantitative



# Results: Quantitative