Autonomous and Robust Time Scale Algorithm for a Swarm of Nanosatellites

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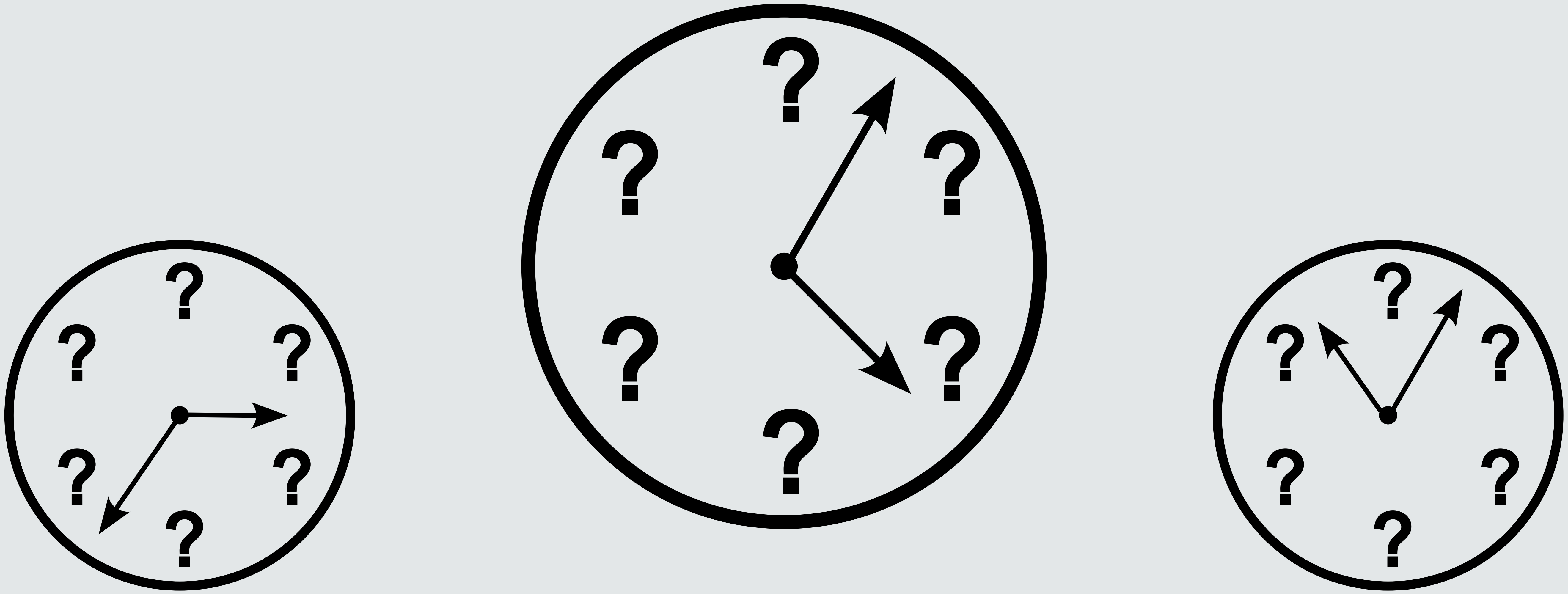
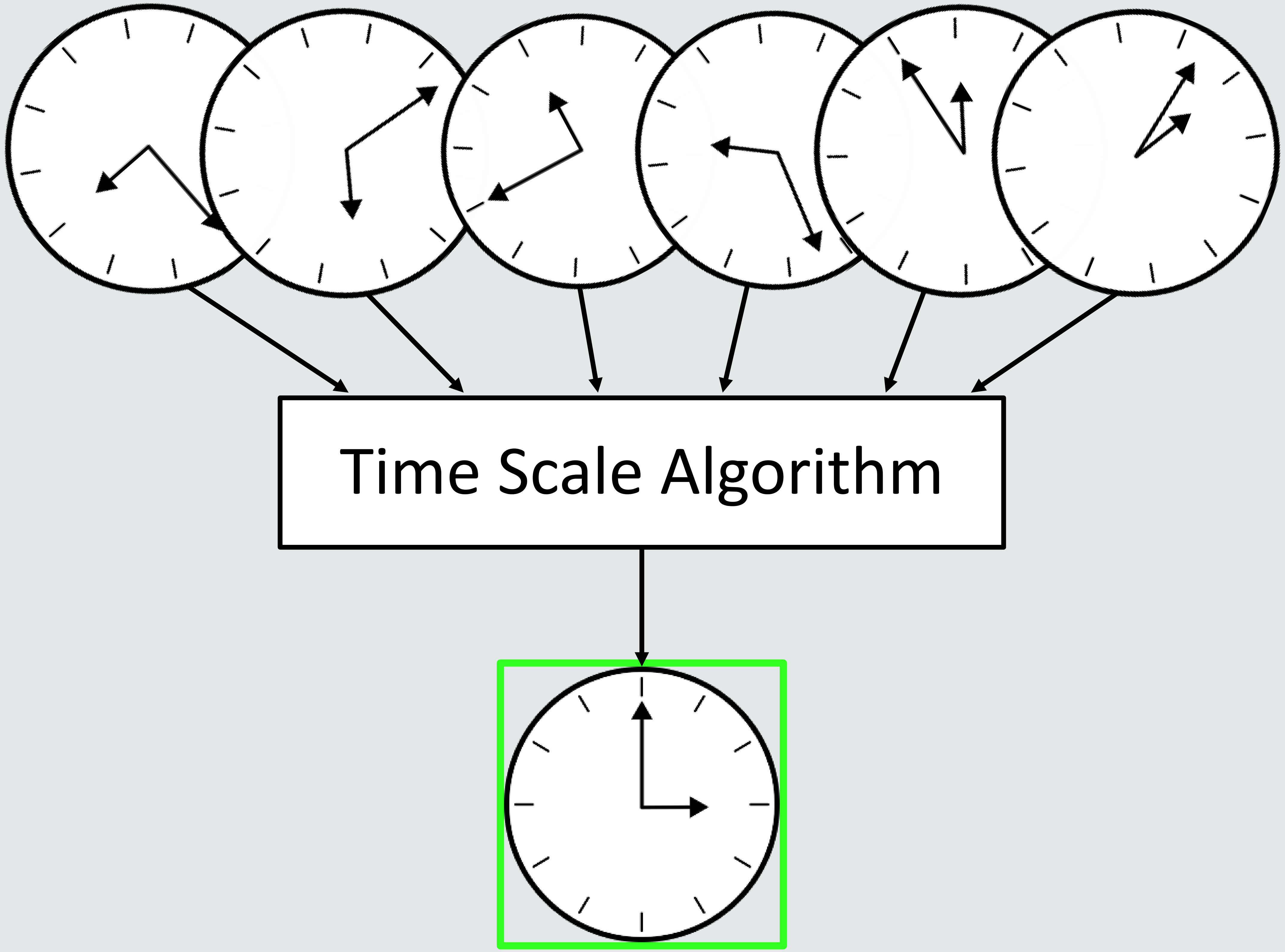


What Time is it?

How can we track time?

- Counting:
  - Sunrises (days)
  - Grains of sand (seconds)
  - Oscillations of an excited atom (picoseconds)
  - No two clocks will count the same time! – Time Scale

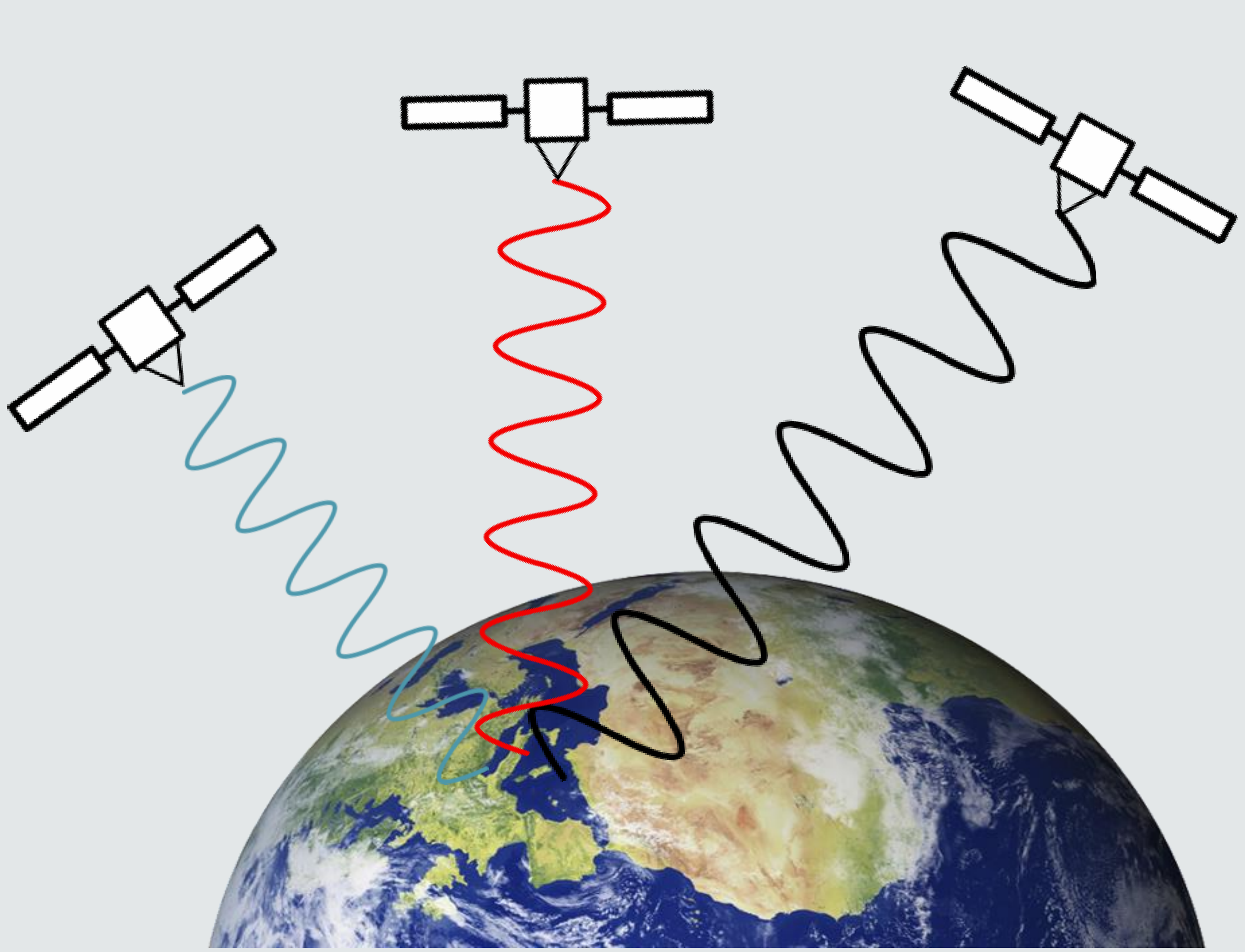
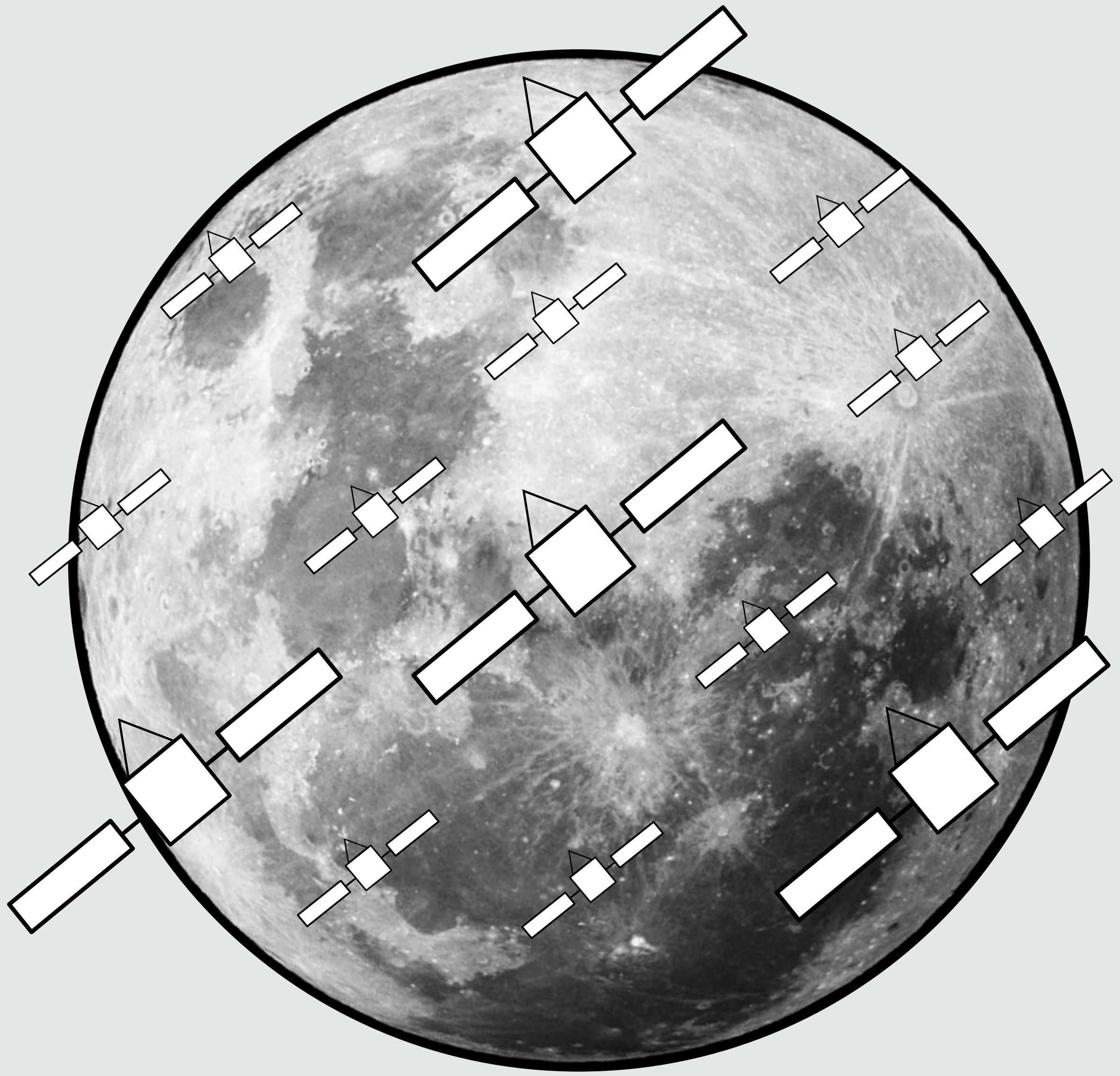


Content

1. Context
2. Contribution 1: ATST
3. Contribution 2: MCRB
4. Results
5. Conclusion

Questions Answered

1. Why is timing important?
2. How can we make a robust Time Scale?
3. How can we quantify the improvement?
4. How does the ATST perform?
5. What was achieved and how can it be improved?

Why is timing important?

And what problems need to be addressed?

Satellite Swarm

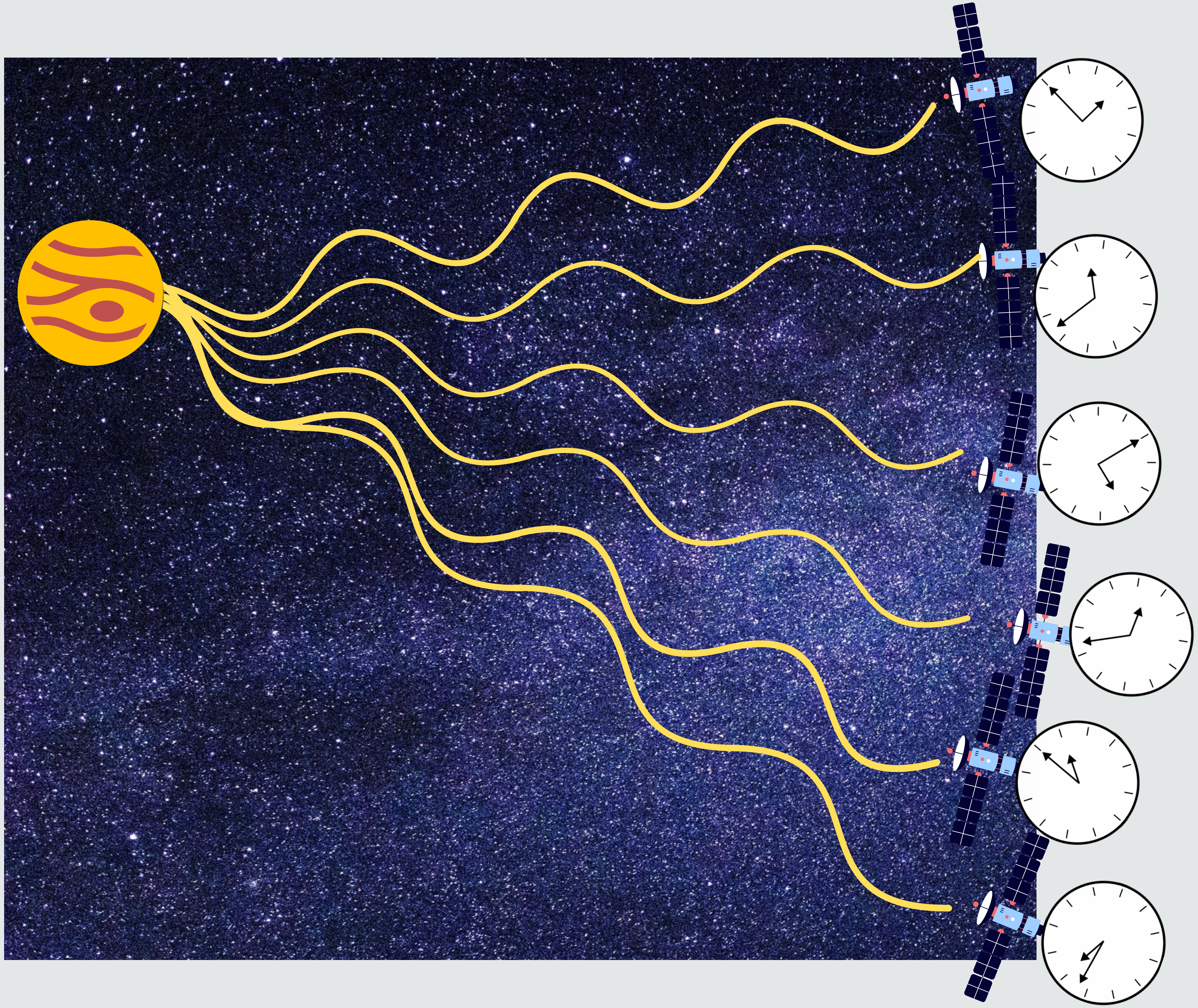
Common radio source, equal speed

Small differences in Time of arrival

More anomalies in space

Robust time scale

Stable and continuous

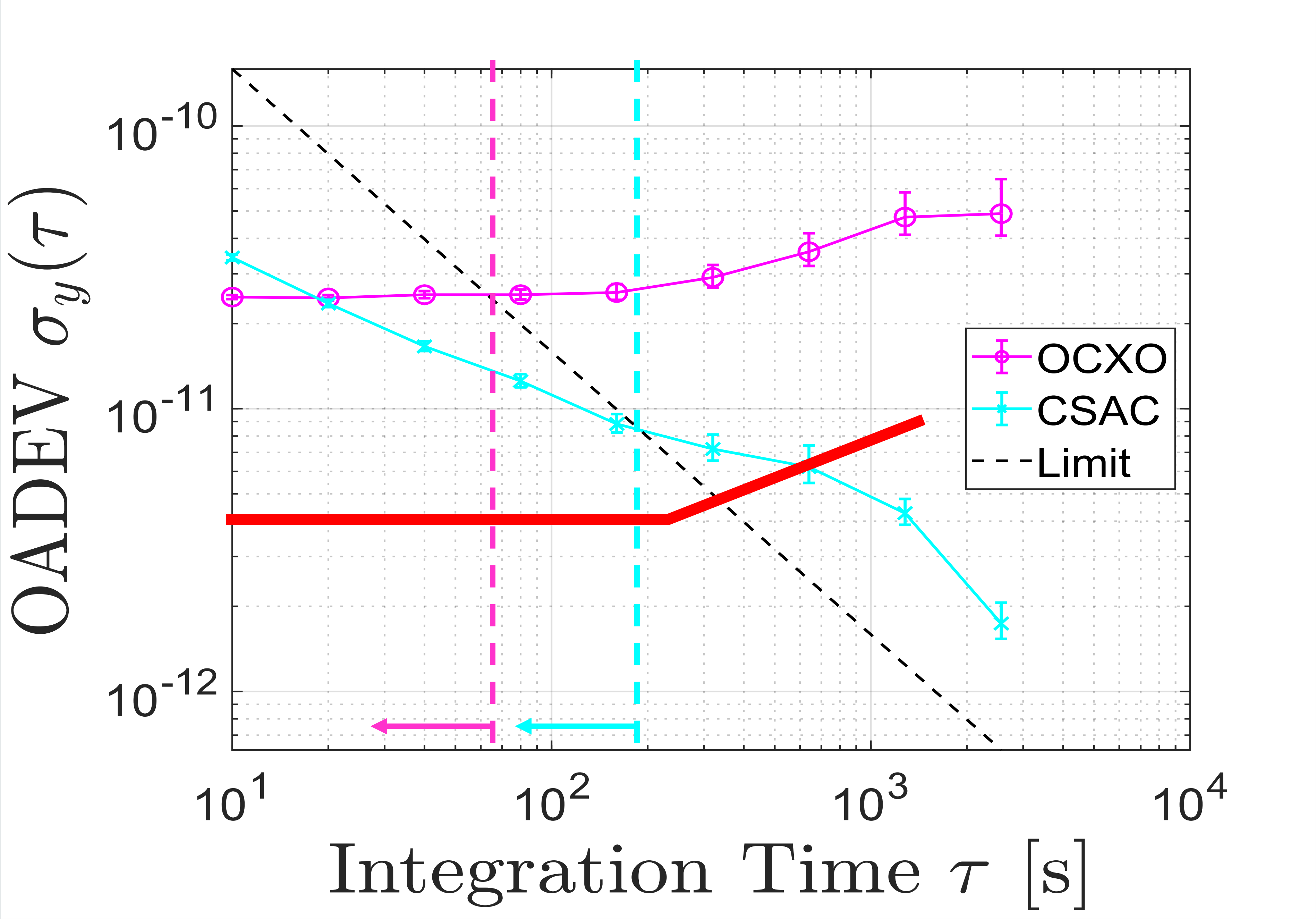


Frequency Stability

- Overlapping Allan Deviation (OADEV)
- Stability constraint:

$$f_{obs} = 100 \text{ MHz}$$
$$\tau\sigma_y(\tau) \leq \frac{1}{2\pi f_{obs}}$$

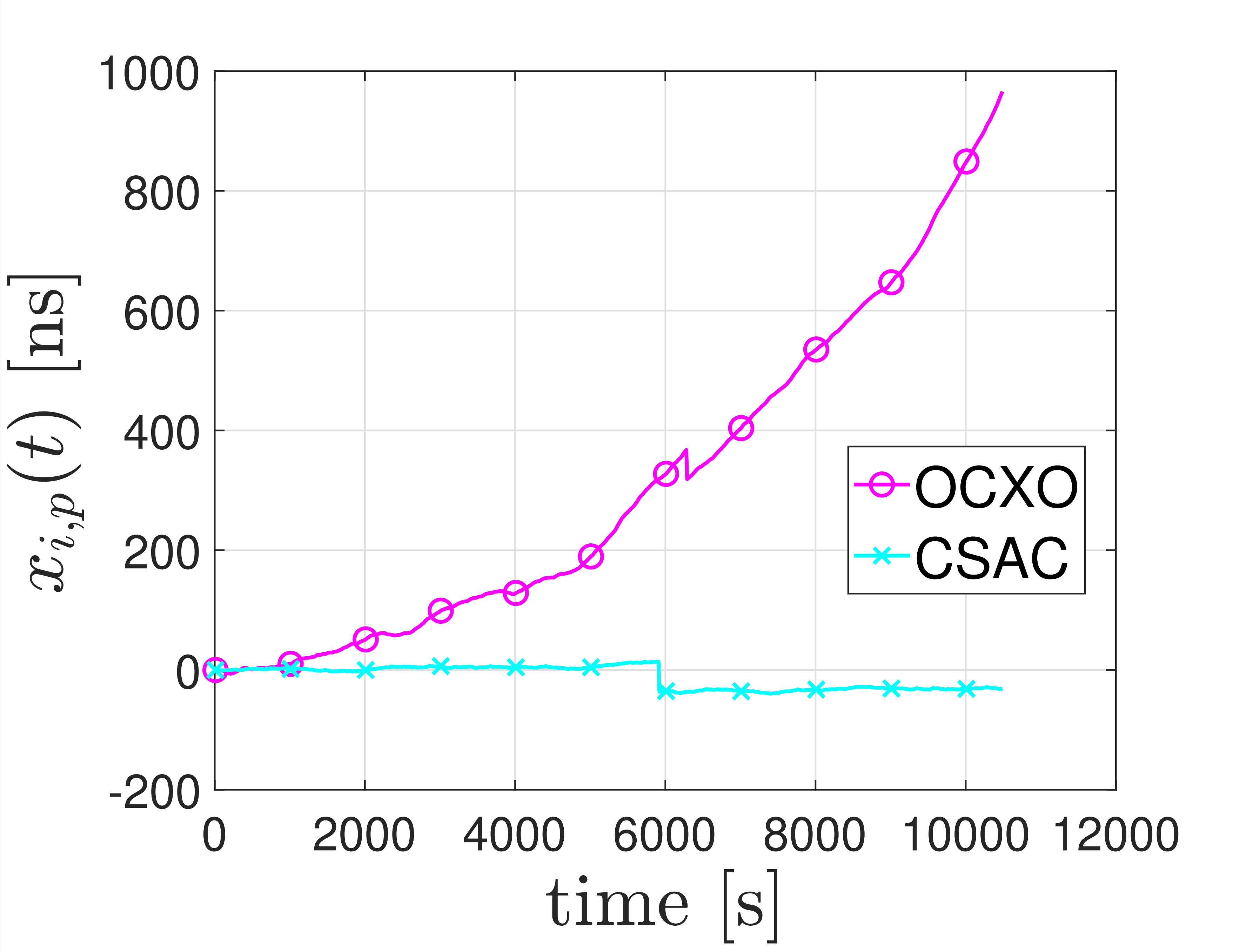
- CSAC provides larger operational range
- Time scale improves range



Oven Controlled Crystal Oscillator (OCXO)
Chip Scale Atomic Clock (CSAC)

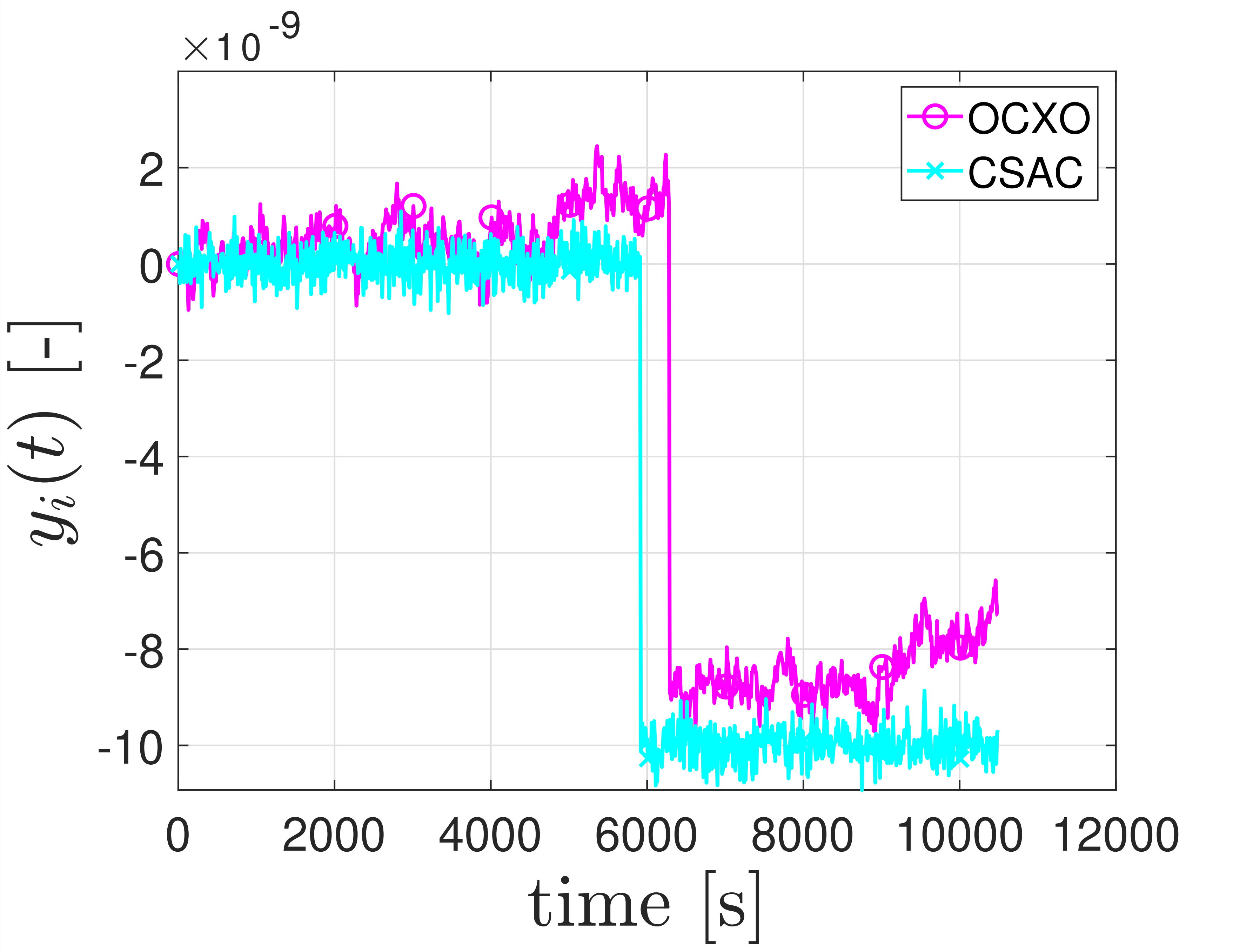
Anomalies and Continuity

- Phase jumps
  - –Occur in timing devices
  - –Broken phase continuity
  - –Broken frequency continuity



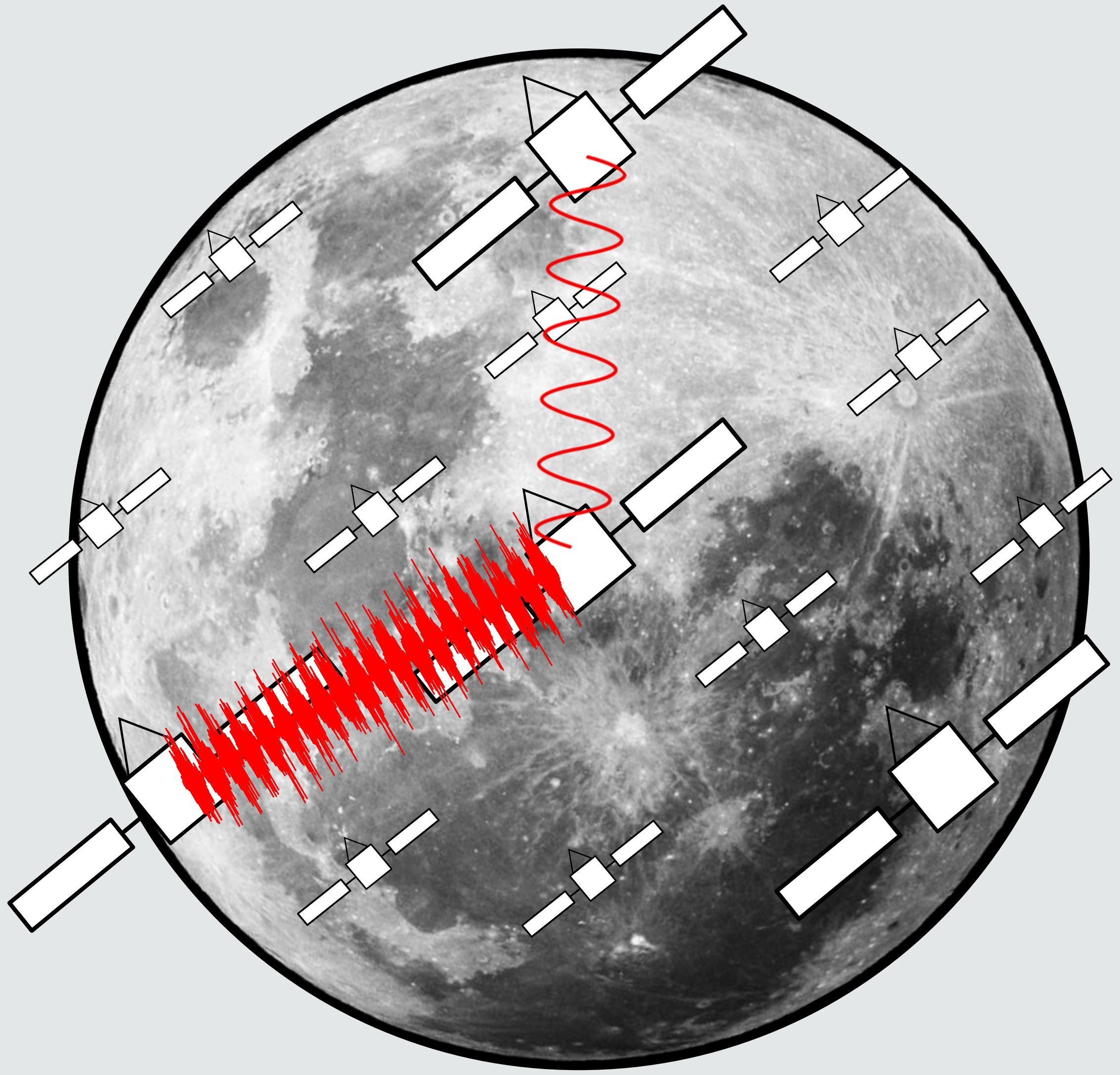
Anomalies and Continuity

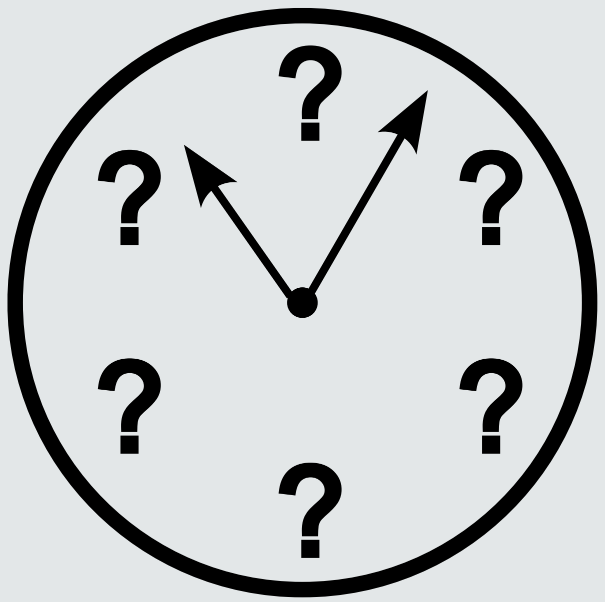
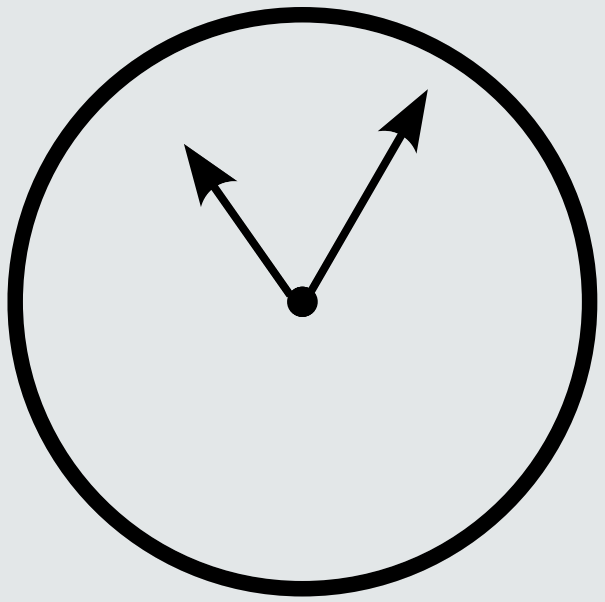
- Frequency jumps
  - Occur in timing devices
  - Increased phase errors
  - Broken frequency continuity



Anomalies and Continuity

- Measurement Link anomalies
  - –Jumps in time scale
- Changes in clock availability
  - –Jumps in time scale

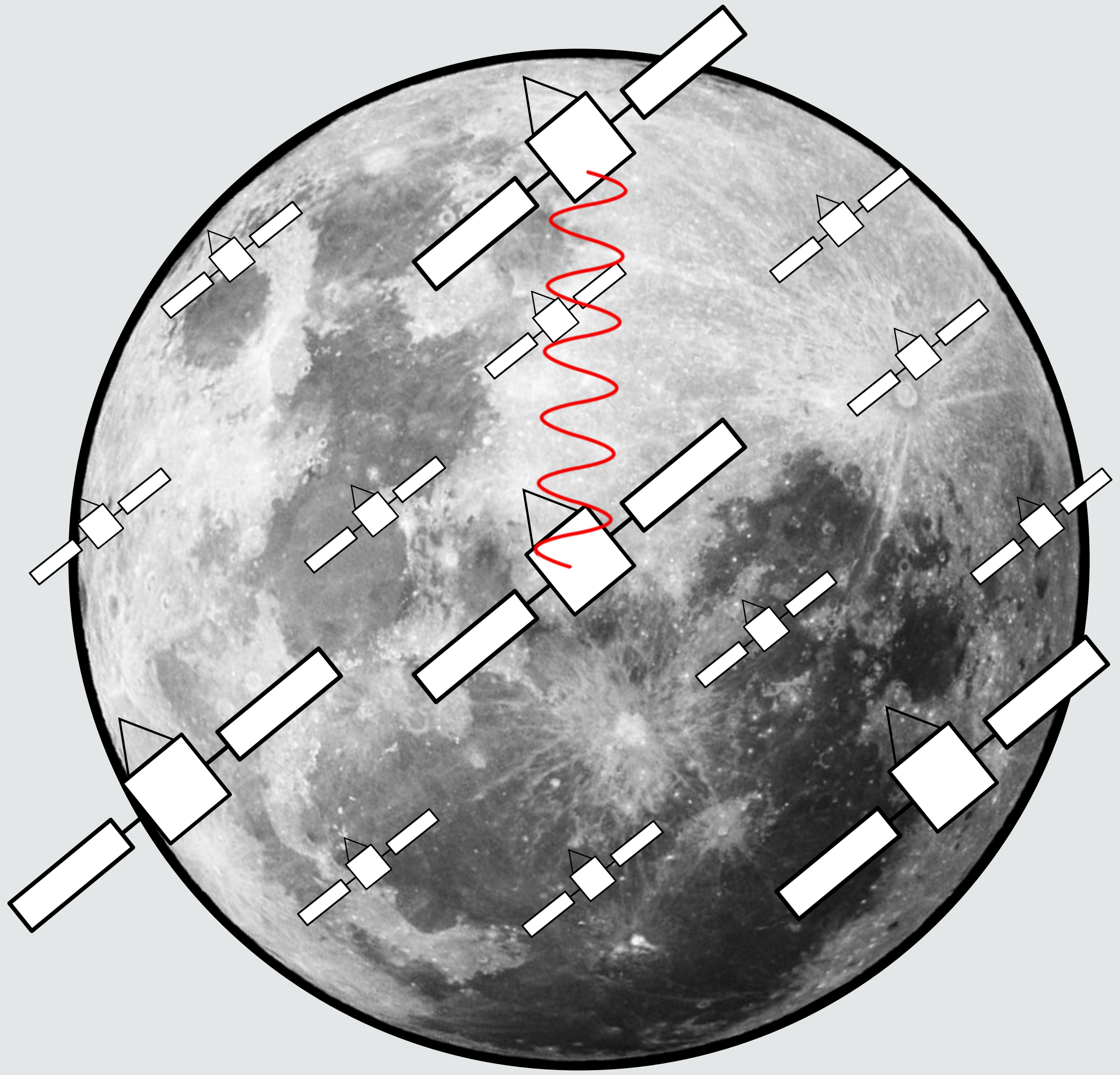


How can we generate a Time Scale?

Notations

$$V(t) = V_0(t)\big(1 + \alpha(t)\big)\cos\big(2\pi\nu_0 t + \phi(t)\big)$$



- Absolute time: $x_{i,p}(t) = h_i(t) - h_p(t) = \frac{\phi(t)}{2\pi\nu_0}$ [s]
- Fractional Frequency: $y_{i,p}(t) = \frac{dx_{i,p}(t)}{dt}$ [s/s]
- Frequency Drift: $d_{i,p}(t) = \frac{dy_{i,p}(t)}{dt}$ [s/s^2]
- Phase measurements: $z_{j,i}(t) = x_{j,p}(t) - x_{i,p}(t) + n_{j,i}(t)$ [s]

Clock Predictions

- Clock i is predicted, assuming constant frequency:

$$\hat{x}_{i,E}(t) = x_{i,E}(t-\tau) + \tau y_{i,E}(t-\tau)$$

$$\hat{y}_{i,E}(t) = y_{i,E}(t-\tau)$$

- Frequency estimation: Smoothing function

Basic Time Scale Equation (BTSE)

- Time scale defined as a reference time: $h_E(t)$

$$x_{i,E}(t) = h_i(t) - h_E(t) = \sum_{j=1}^{N} w_j(t)\left(\hat{x}_{j,E}(t) - z_{j,i}(t)\right)$$

$$x_{i,E}(t) = \sum_{j=1}^{N} w_j(t) r_{j,i}(t)$$

$$x_{i,E}(t) = x_{i,p}(t) + \sum_{j=1}^{N} w_j(t)\left(\hat{x}_{j,E}(t) - x_{j,p}(t)\right) + \sum_{j=1}^{N} w_j(t) n_{j,i}(t)$$

Weights

- Inverse of prediction error (AT1/AT2)

$$w_j(t) = \frac{1}{\left(x_{j,E}(t) - \hat{x}_{j,E}(t)\right)^2}$$

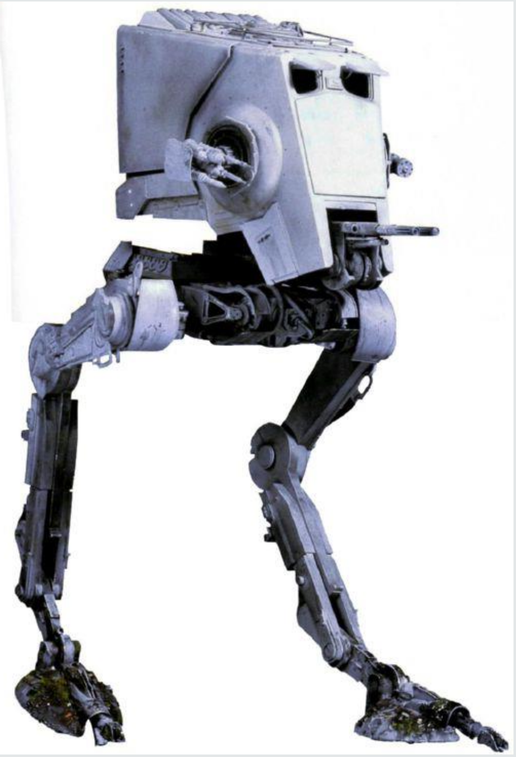
$$\epsilon_j(t) = \left|x_{j,E}(t) - \hat{x}_{j,E}(t)\right|, \qquad w_j(t) = \min\left(\frac{4}{N}, \frac{\epsilon_j^2(t)}{\sum_{l=1}^{N} \epsilon_j^2(t)}\right)$$

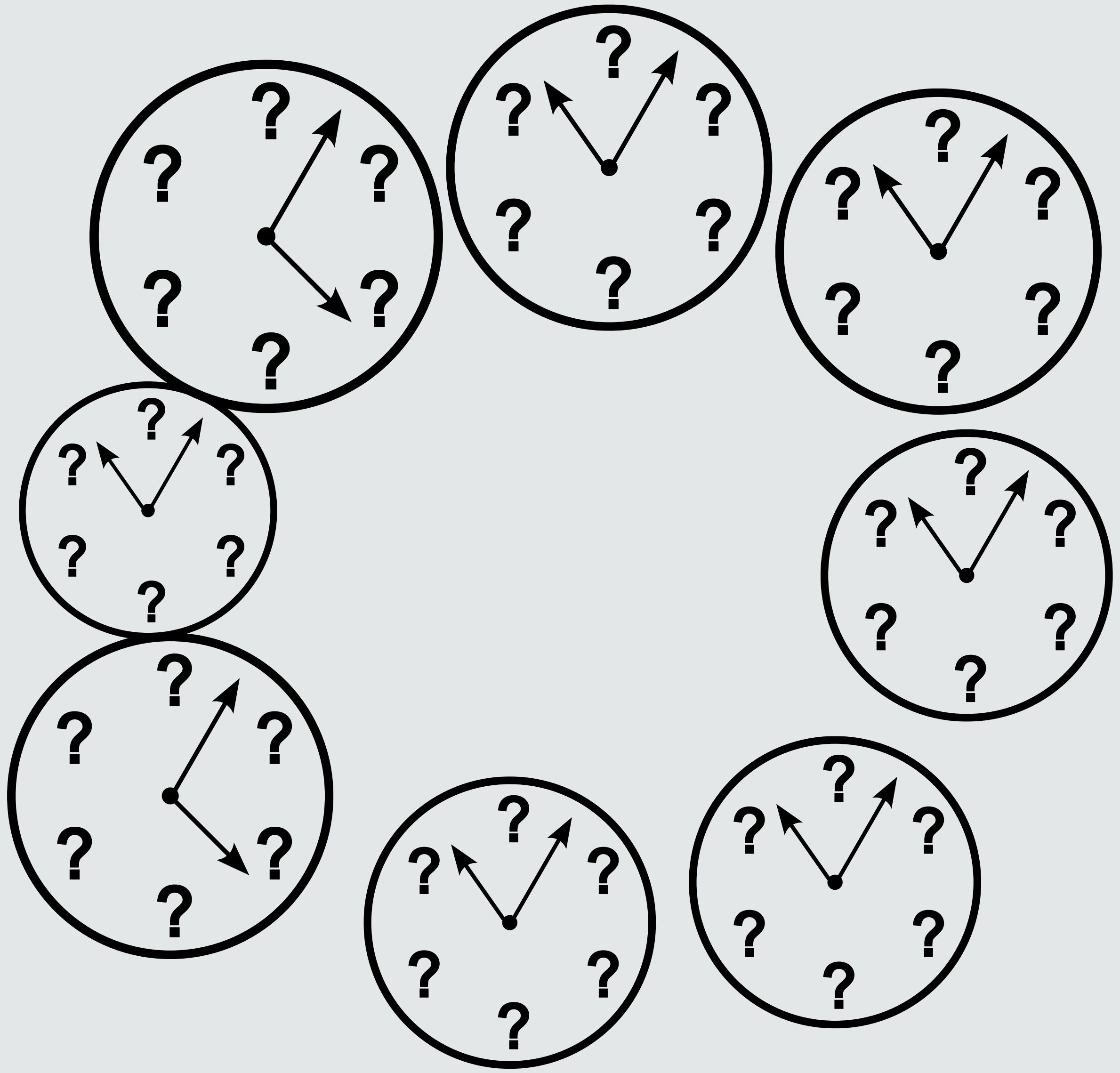
$$x_{i,E}(t) = \sum_{j=1}^{N} w_j(t-\tau)\left(\hat{x}_{j,E}(t) - z_{j,i}(t)\right)$$

How to update the weights?

Anomaly / Time Scale	Phase jump	Frequency jump	Link Anomaly	Missing Data
AT1/AT2	$w_j(t)$ adapted when $\epsilon_j(t) > S_1$ Recompute $x_{i,E}(t)$	$x_{i,E}(t - L\tau)$ recomputed if $\lvert y_{j,E}(t) - \bar{y}_j(t) \rvert > S_2$	Undefined	Not found
Proposed	Automatically update: $w_j(t)$, $x_{j,E}(t)$	Automatically update: $w_j(t)$, $x_{j,E}(t)$	Automatically update: $w_j(t)$, $x_{j,E}(t)$	Set relevant weights to zero [1]

[1] McPhee, Hamish, et al. ‘Exploiting Redundant Measurements for Time Scale Generation in a Swarm of Nanosatellites’. *Proceedings of the 37th Annual European Frequency and Time Forum (EFTF)*, Neuchâtel, Switzerland, 2024.

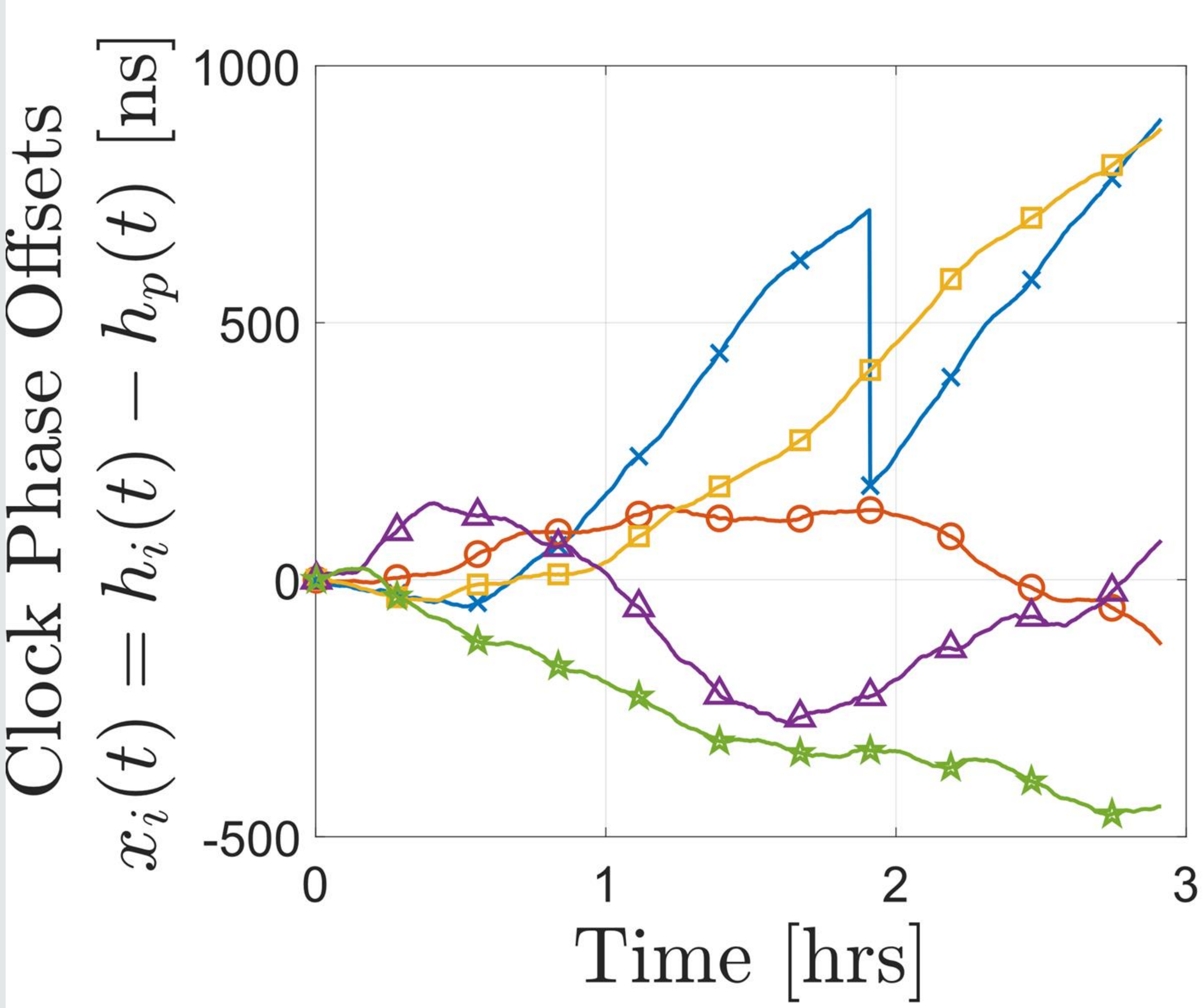


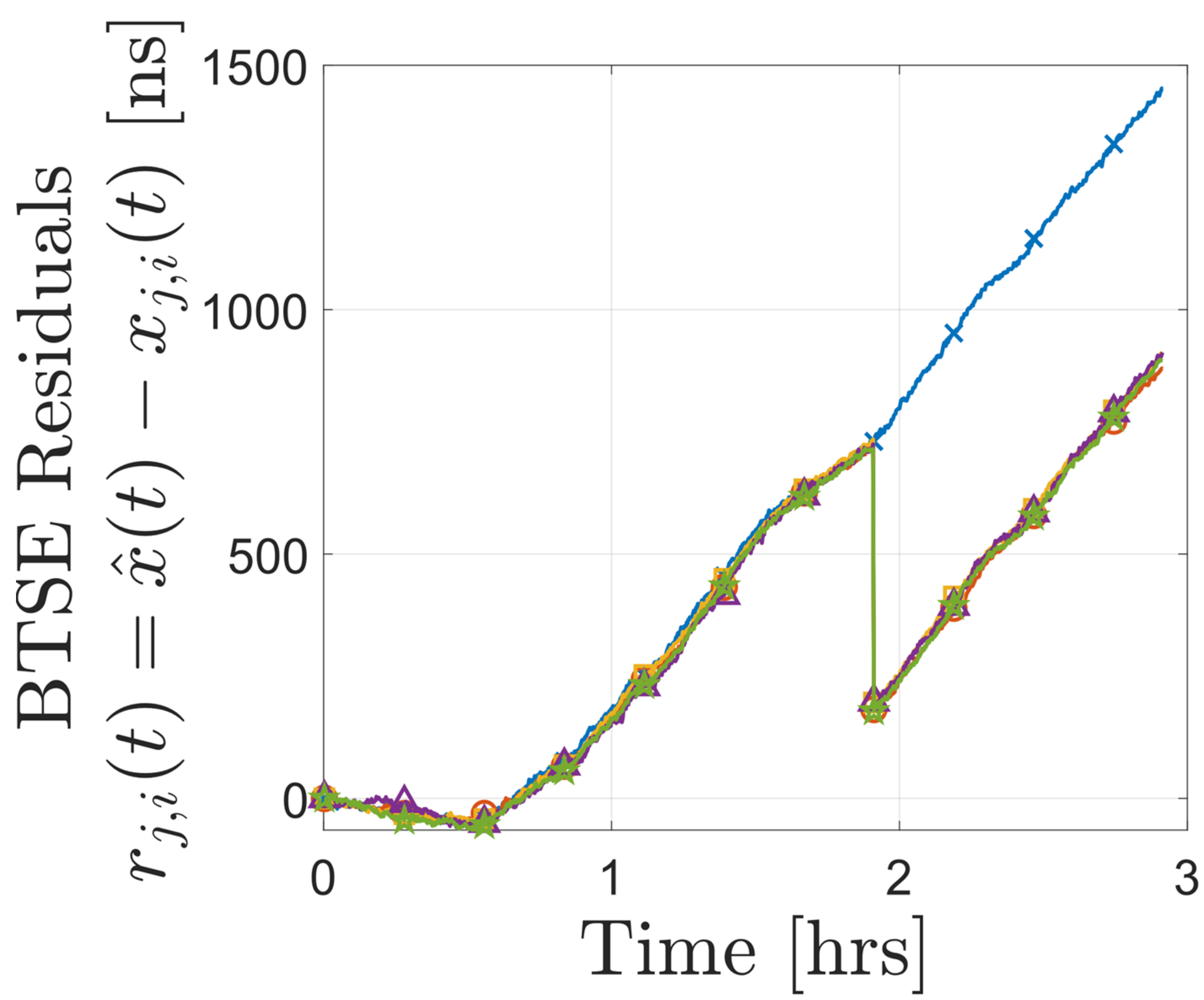


First Contribution:
An Autonomous Time scale using the Student’s T-distribution (ATST)

BTSE residuals

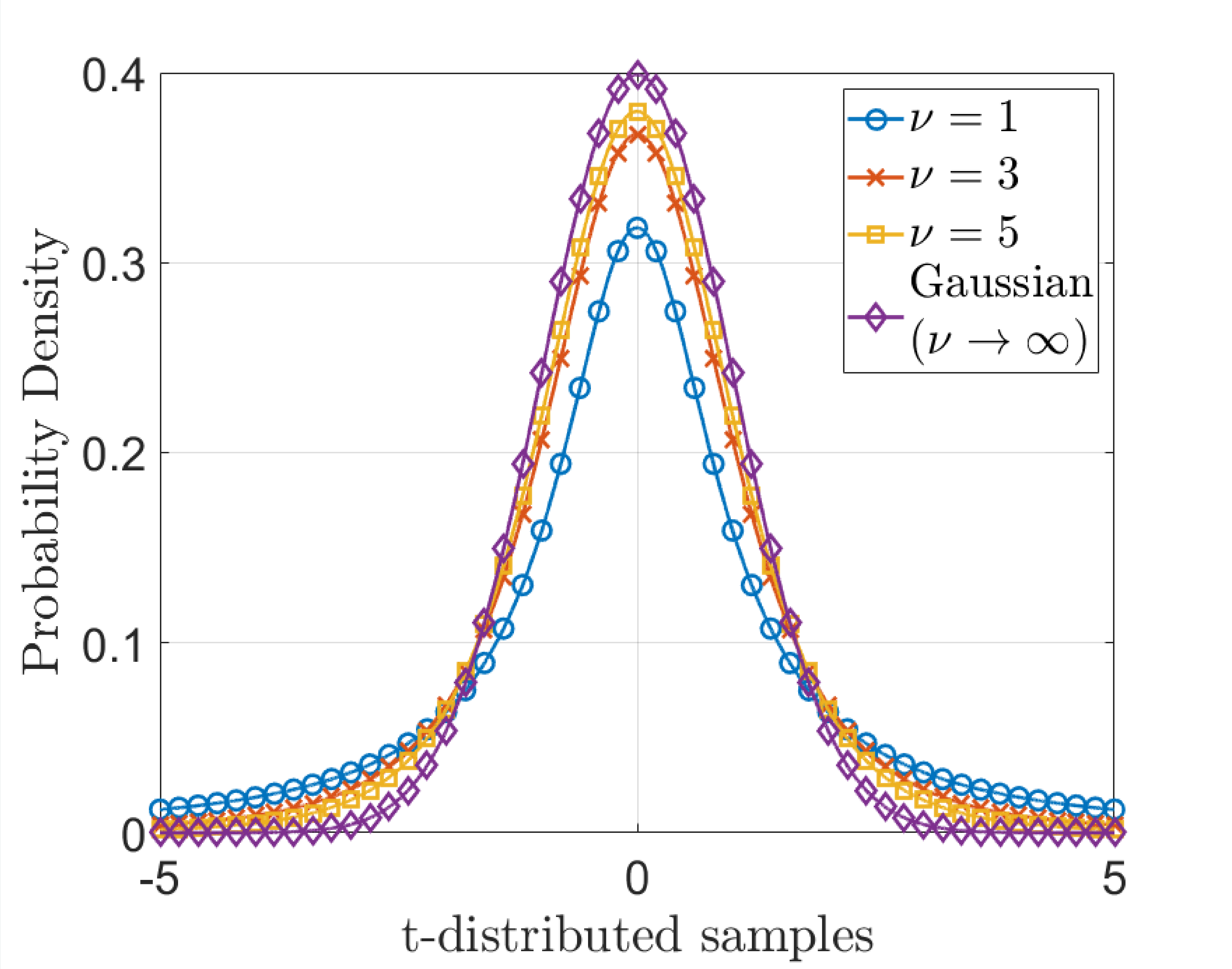
$$x_{i,E}(t) = \sum_{j=1}^{N} w_j(t)\big(\hat{x}_{j,E}(t) - z_{j,i}(t)\big) = \sum_{j=1}^{N} w_j(t) r_{j,i}(t)$$





Student’s t-distribution

$r_{j,i}(t) \sim N\left(x_{i,p}(t), \sigma^2(t)\right)$ $\qquad$ $r_{j,i}(t) \sim T\left(x_{i,p}(t), \sigma^2(t), \nu(t)\right)$



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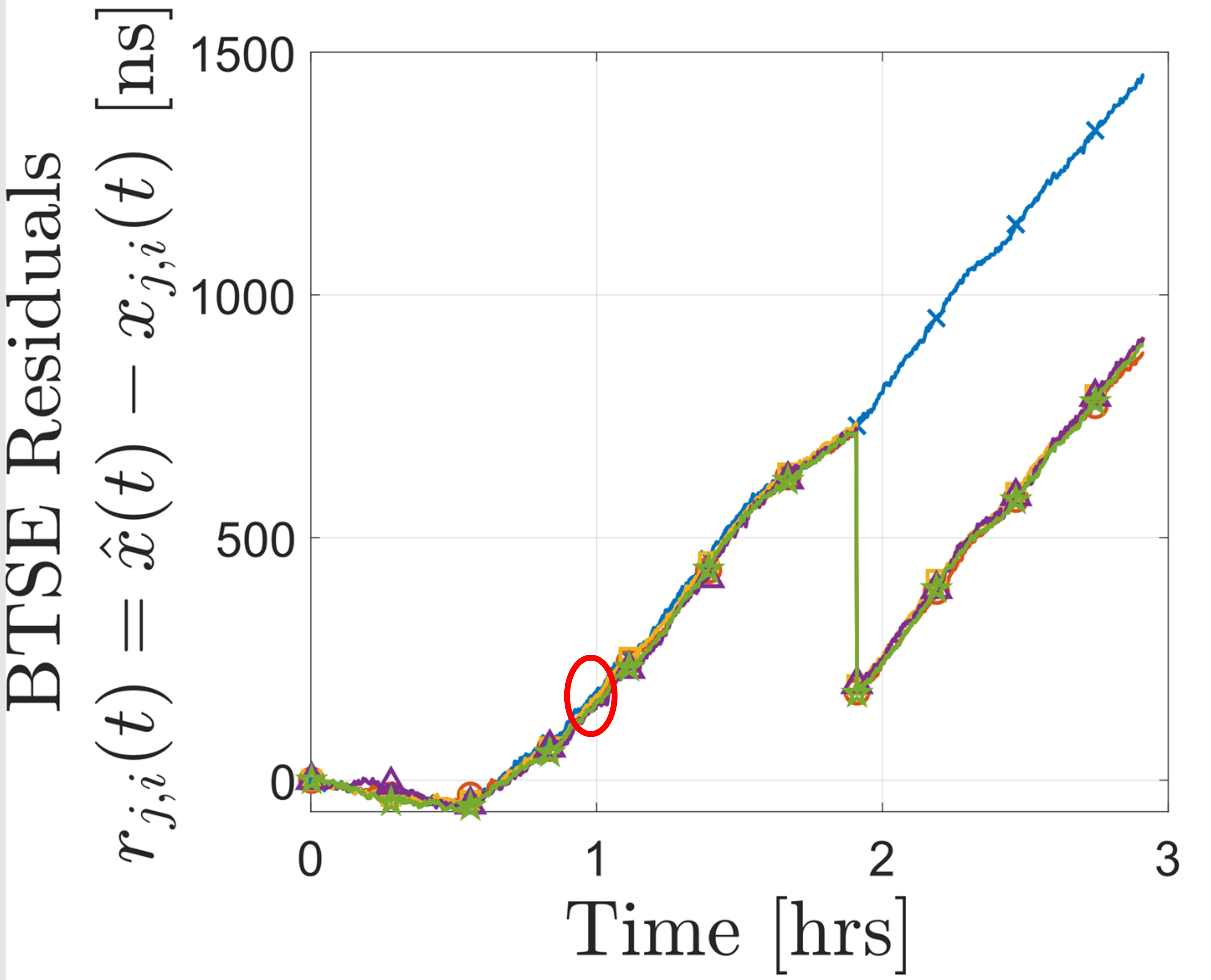

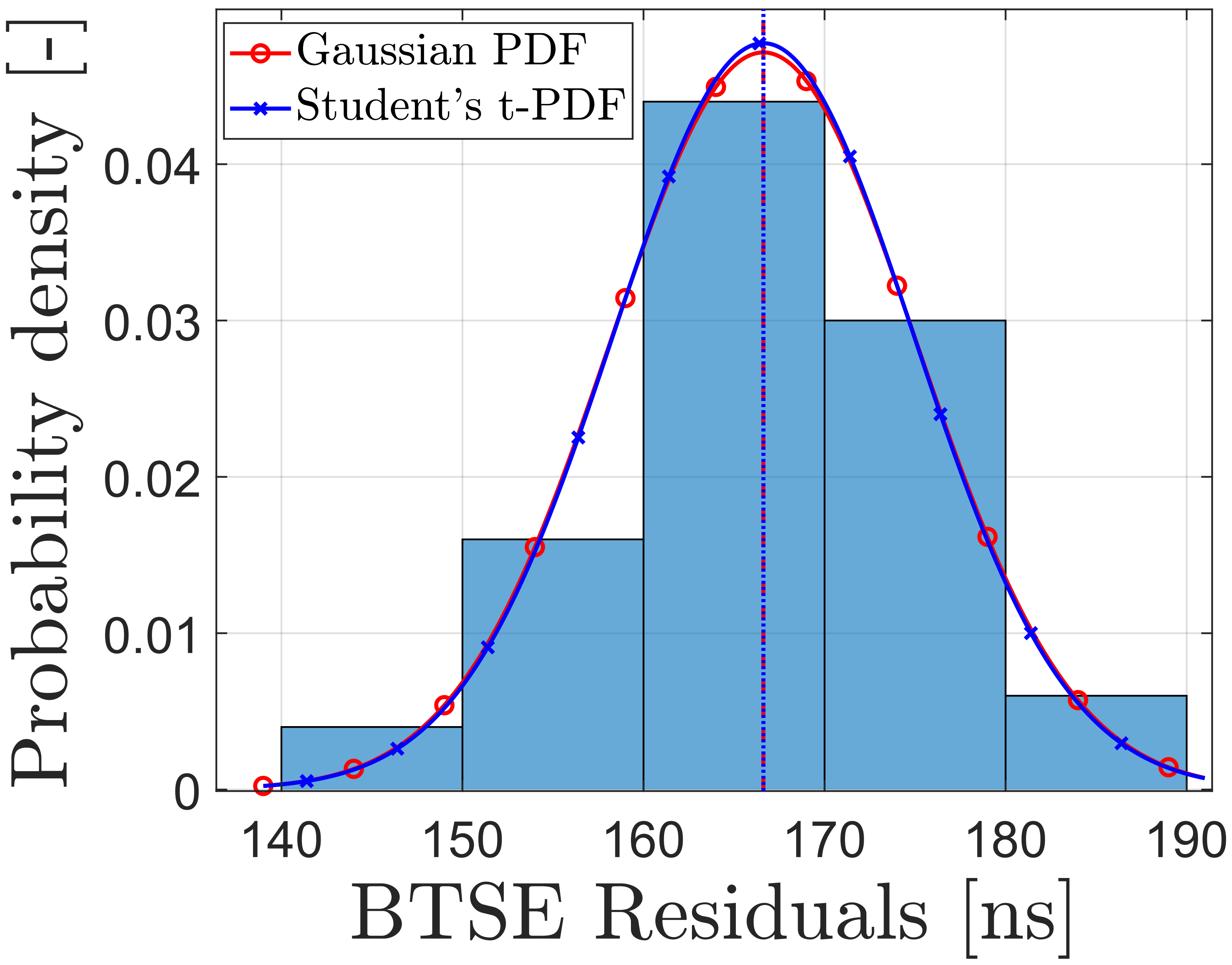

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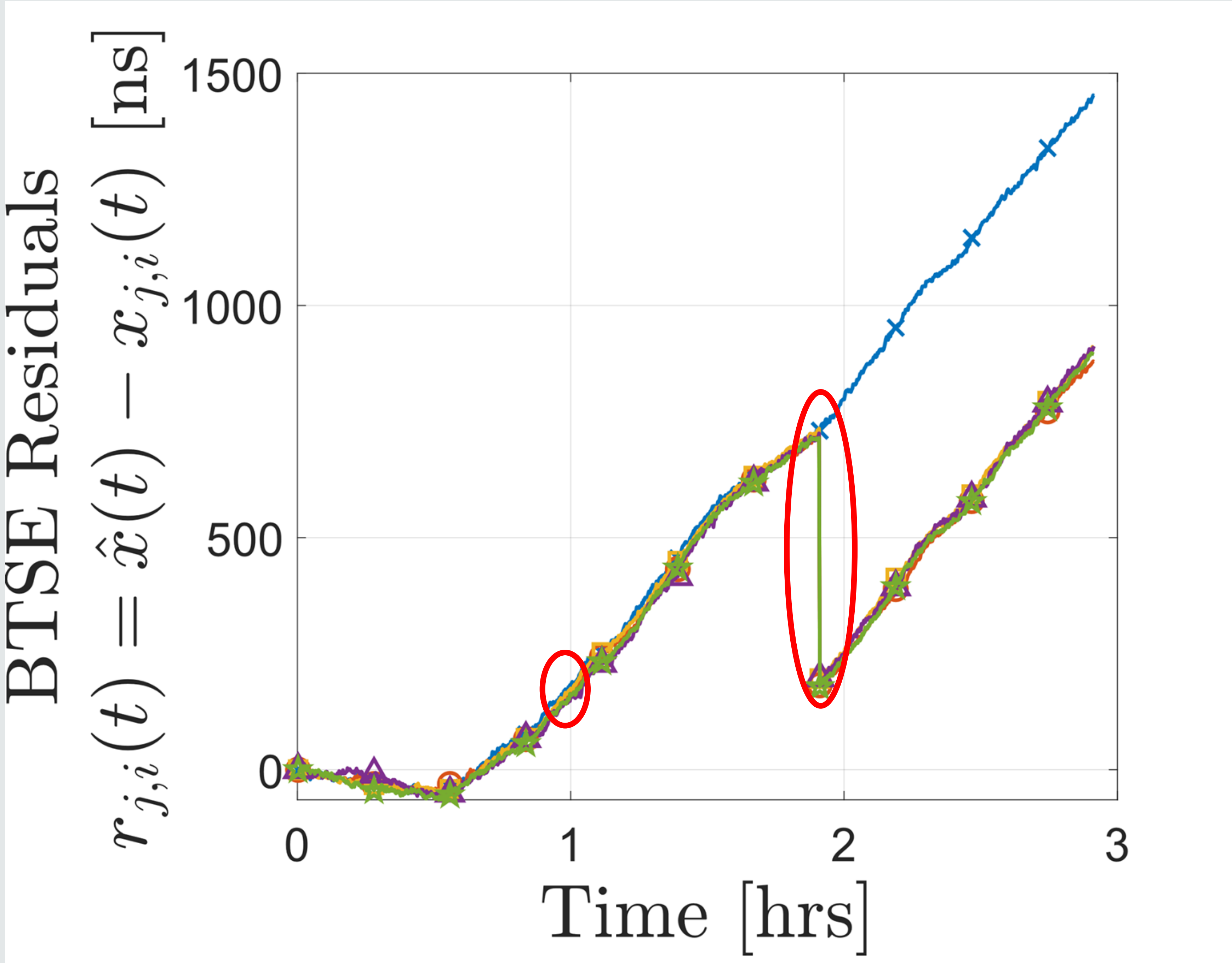
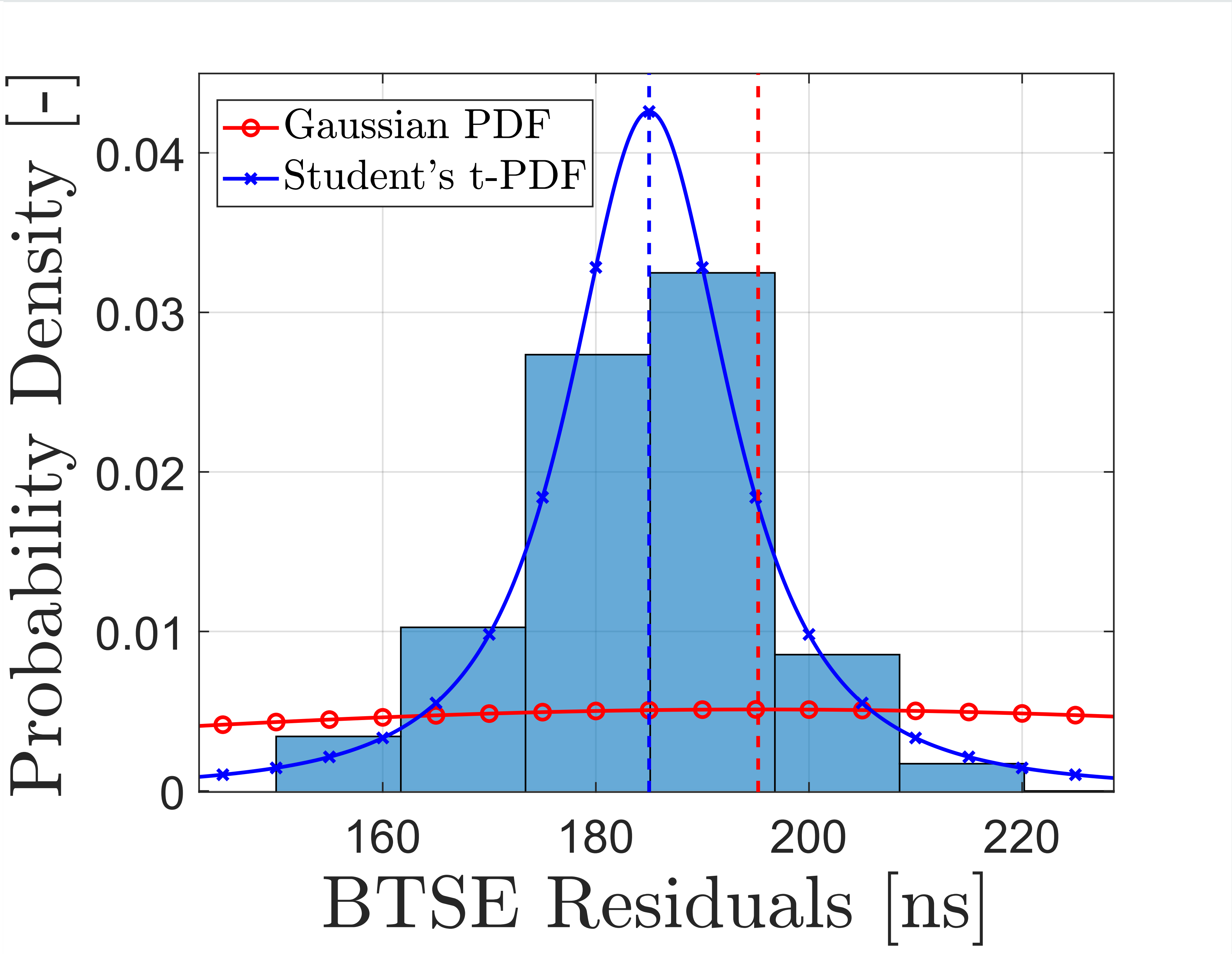




Parameter Estimation

- Gaussian data, Maximum Likelihood Estimator (MLE):

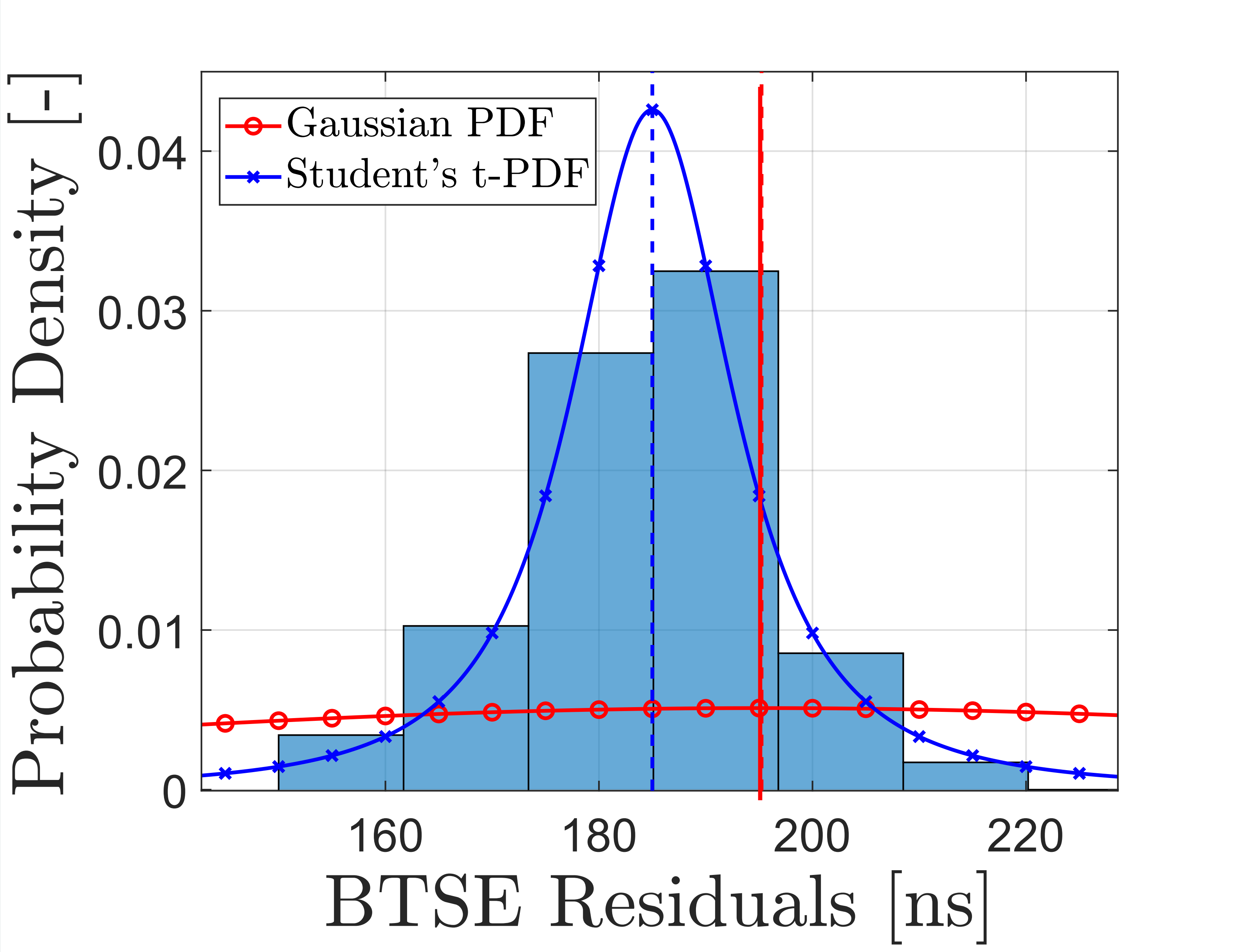
$$\hat{\mu}(t) = \frac{1}{N}\sum_{j=1}^{N} r_{j,i}(t) = x_{i,E}(t)$$

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{j=1}^{N} \left(r_{j,i}(t) - x_{i,E}(t)\right)^2$$

- Student t-distributed data, no closed form MLE

Expectation-Maximization (EM)

- Initialize $\hat{\mu}_0(t)$, $\hat{\sigma}_0^2(t)$, $\hat{\nu}_0(t)$ with Gaussian MLE, then use EM algorithm



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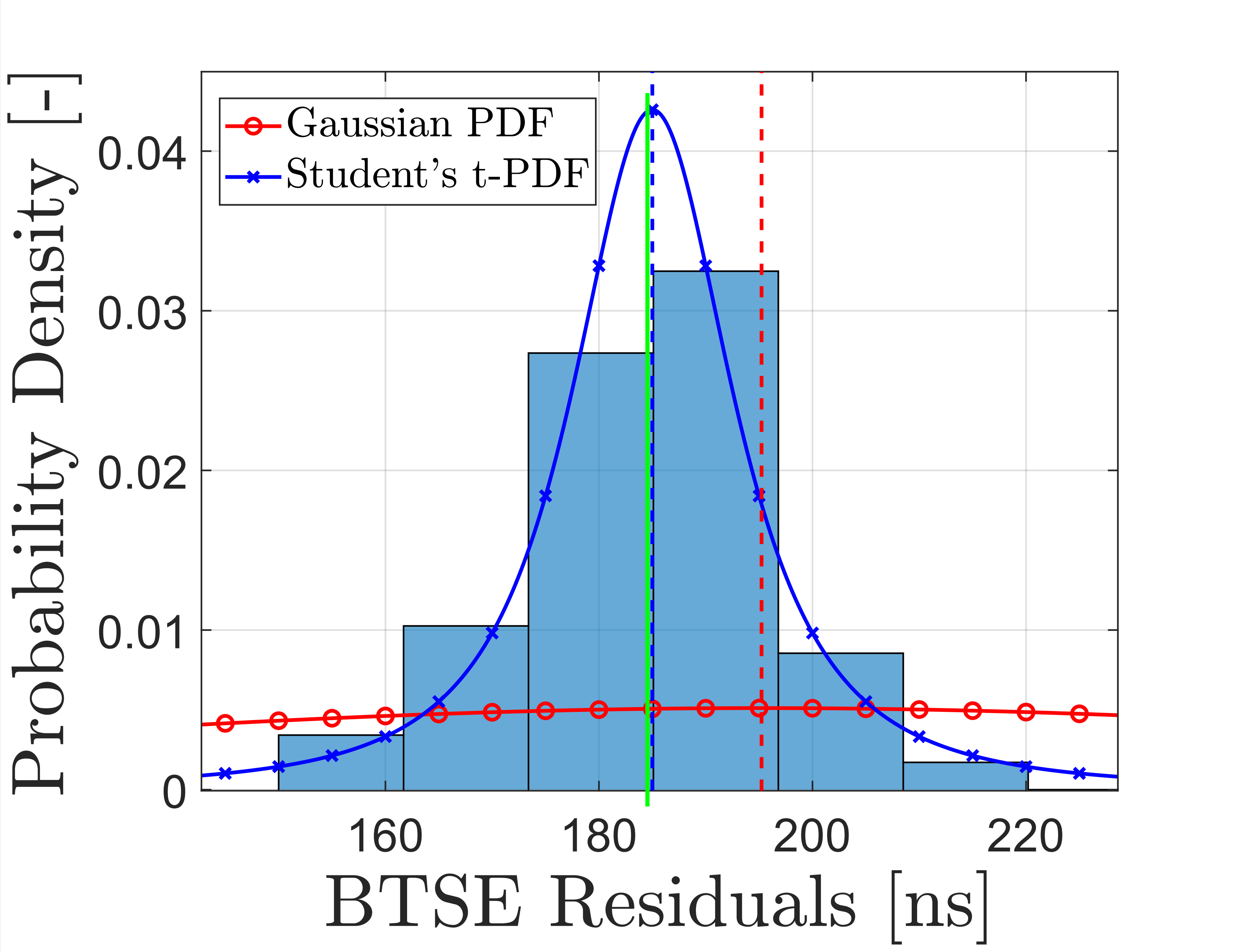


Expectation-Maximization (EM)

- Initialize $\hat{\mu}_0(t)$, $\hat{\sigma}_0^2(t)$, $\hat{\nu}_0(t)$ with Gaussian MLE, then use EM algorithm

while $\hat{\mu}_k - \hat{\mu}_{k-1} > S$

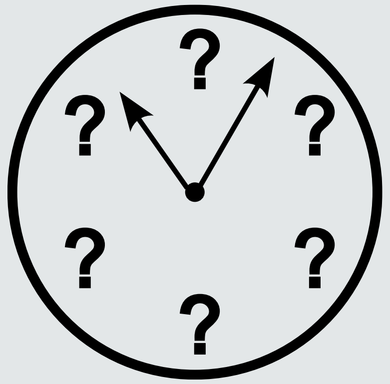
$$u_j(t) = \frac{\hat{\nu}_{k-1} + 1}{\hat{\nu}_{k-1} + \frac{\left(r_{j,i}(t) - \hat{\mu}_{k-1}(t)\right)^2}{\hat{\sigma}_{k-1}^2}}, \qquad w_j(t) = \frac{u_j(t)}{\sum_{l=1}^{N} u_l(t)}$$

$$\hat{\mu}_k(t) = \sum_{j=1}^{N} w_j(t) r_{j,i}(t) = x_{i,E}(t)$$

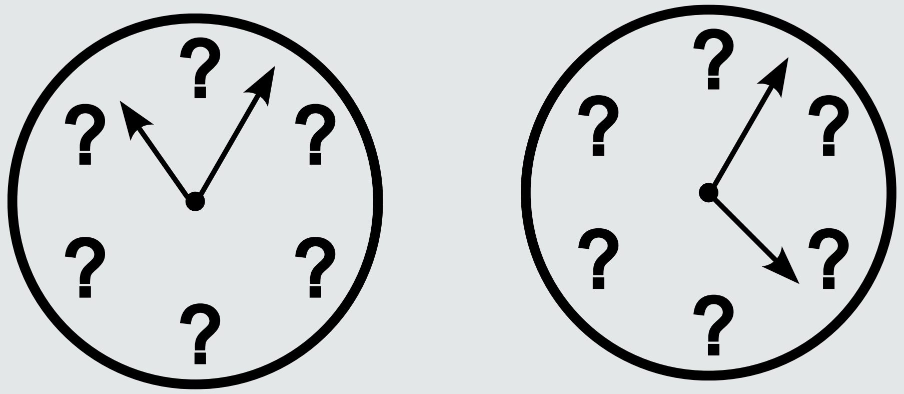
ATST algorithm

At each time t, the ATST steps include [2]:

1. Predict clock phase and frequency

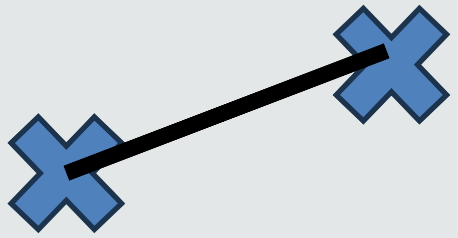


2. Obtain measurements – BTSE residuals



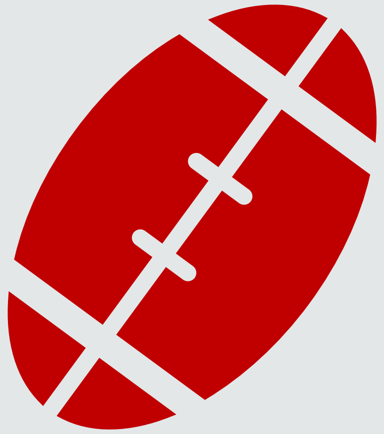
3. Recursive EM algorithm – Equivalent to BTSE
    - Weights adapted for anomalous clocks

4. Frequency estimation – Equivalent to AT1 (tuning)



[2] McPhee, Hamish, et al. ‘A Robust Time Scale for Space Applications Using the Student’s t-Distribution’. *Metrologia*, vol. 61, no. 5, Sept. 2024, p. 055010.

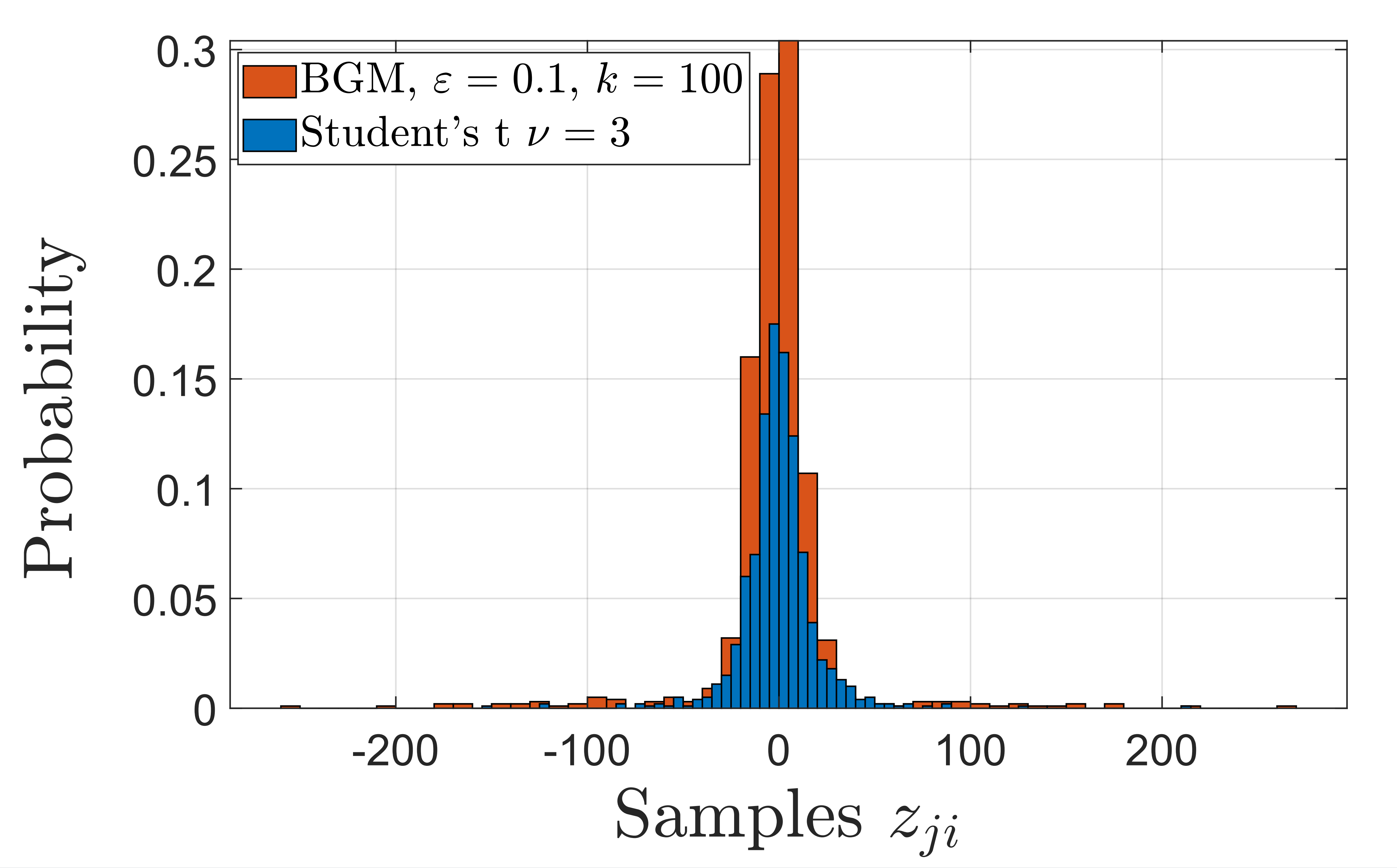
How good is ATST in theory?



Second Contribution: Misspecified Cramér-Rao Bounds applied to a Robust Time Scale

Heavy-tailed distributions

- Student's t-distribution
  - Number of degrees of freedom: ν
- Bimodal Gaussian Mixture (BGM)
  - Proportion of anomalous data: ε
  - Anomalies with inflated variance: $k\sigma^2$



What is a Cramér-Rao Bound?

- Theoretical lower limit of estimation error for parameter vector $\boldsymbol{\theta}$

$$\mathbf{CRB}_{\boldsymbol{\theta}} = -\frac{1}{N} E_p\left[\frac{\partial^2 \log\big(p(\mathrm{z_i};\boldsymbol{\theta})\big)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T}\right] = \frac{1}{N} E_p\left[\frac{\partial \log\big(p(\mathrm{z_i};\boldsymbol{\theta})\big)}{\partial\boldsymbol{\theta}}\frac{\partial \log\big(p(\mathrm{z_i};\boldsymbol{\theta})\big)}{\partial\boldsymbol{\theta}}^T\right]$$

- MSE of the MLE is known to asymptotically approach the CRB

- Closed-form expressions for Gaussian and Student’s t-distributions

$$\mathrm{CRB}_{\mu} = \frac{\sigma^2}{N}, \qquad \mathrm{CRB}_{\mu_T} = \left(\frac{\nu+3}{\nu+1}\right)\left(\frac{\nu-2}{\nu}\right)\frac{\sigma^2}{N}$$

Misspecified Cramér-Rao Bound

- Misspecified = Incorrect assumption on statistical model

$$\mathbf{MCRB}_{\theta}(\mathrm{p}||\mathrm{q}) = \frac{1}{N}\mathbf{A}(\boldsymbol{\theta}_p)^{-1}\mathbf{B}(\boldsymbol{\theta}_p)\mathbf{A}(\boldsymbol{\theta}_p)^{-1}, \qquad \mathbf{A}(\boldsymbol{\theta}_p) = E_p\left[\frac{\partial^2 \log(q(\mathrm{z_i};\boldsymbol{\theta}))}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T}\right]\Bigg|_{\boldsymbol{\theta}_p},$$

$$\mathbf{B}(\boldsymbol{\theta}_p) = E_p\left[\frac{\partial \log(q(z_i;\boldsymbol{\theta}))}{\partial\boldsymbol{\theta}}\frac{\partial \log(q(z_i;\boldsymbol{\theta}))}{\partial\boldsymbol{\theta}}^T\right]\Bigg|_{\boldsymbol{\theta}_p}, \qquad \boldsymbol{\theta}_p = \min_{\boldsymbol{\theta}} \mathrm{KLD}(p||q)$$

True / Assumed	Gaussian	BGM	Student's t
Gaussian	CRB ✓	MCRB ✓*	MCRB ✓*
BGM	MCRB ?	CRB ✓	MCRB ?
Student's t	MCRB ?	MCRB ?	CRB ✓

* Derived during my PhD

Results – Derived Bounds

General derivation of $\boldsymbol{\theta}_p = \left[\mu_p, \sigma_p^2\right]^T = \left[E_p[z_i], \mathrm{var}_p[z_i]\right]^T$

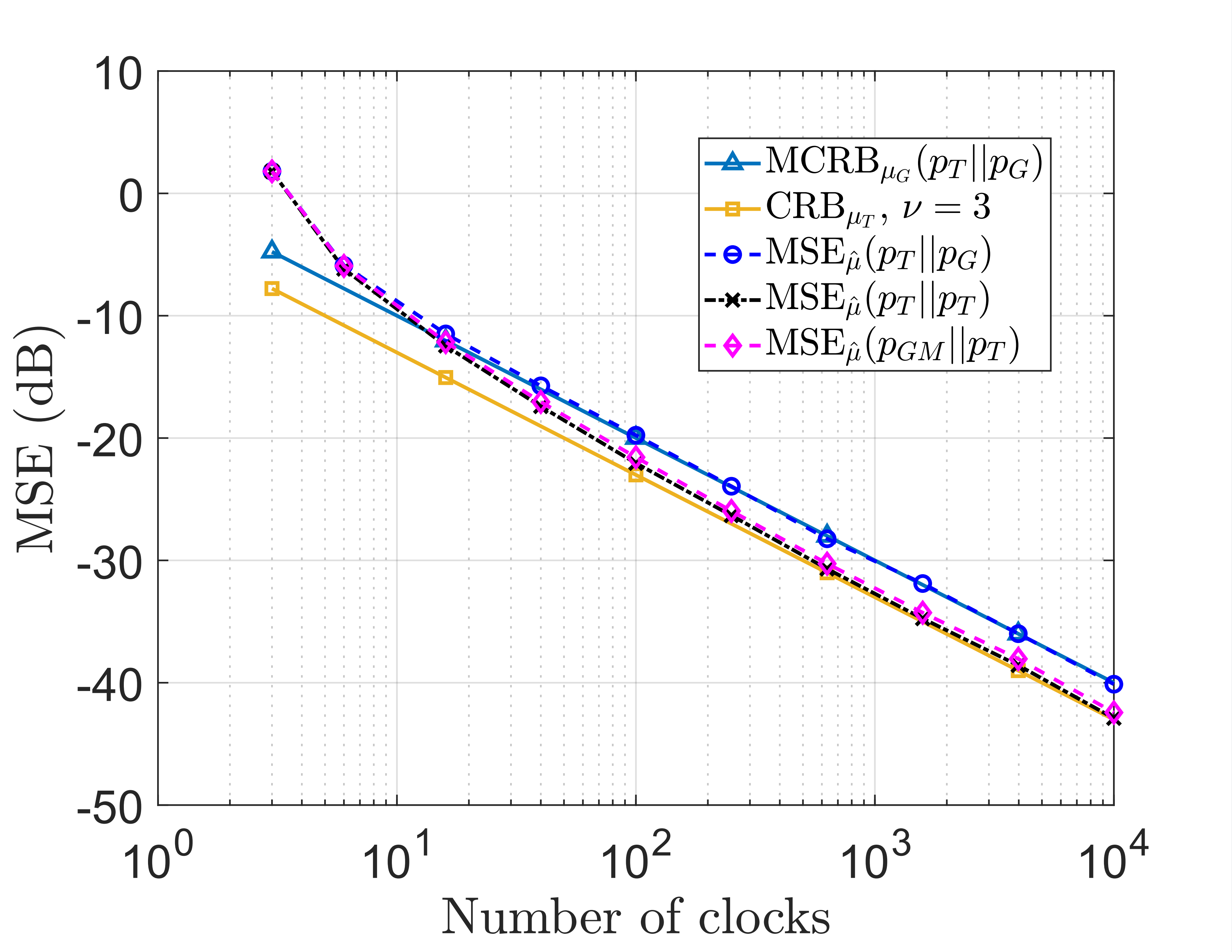
$$\mathbf{MCRB}_{\boldsymbol{\theta}}(p_T||p_G) = \begin{bmatrix} \frac{\sigma_{p_T}^2}{N} & 0 \\ 0 & \left(\frac{\nu-1}{\nu-4}\right)\frac{2\sigma_{p_T}^4}{N} \end{bmatrix}$$

$$\mathbf{MCRB}_{\boldsymbol{\theta}}(p_{GM}||p_G) = \begin{bmatrix} \frac{\sigma_{p_{GM}}^2}{N} & 0 \\ 0 & \left(\frac{Q(\varepsilon(k-1))}{2(\varepsilon(k-1)+1)^2}\right)\frac{2\sigma_{p_{GM}}^4}{N} \end{bmatrix}$$

MCRB Student's t-distribution

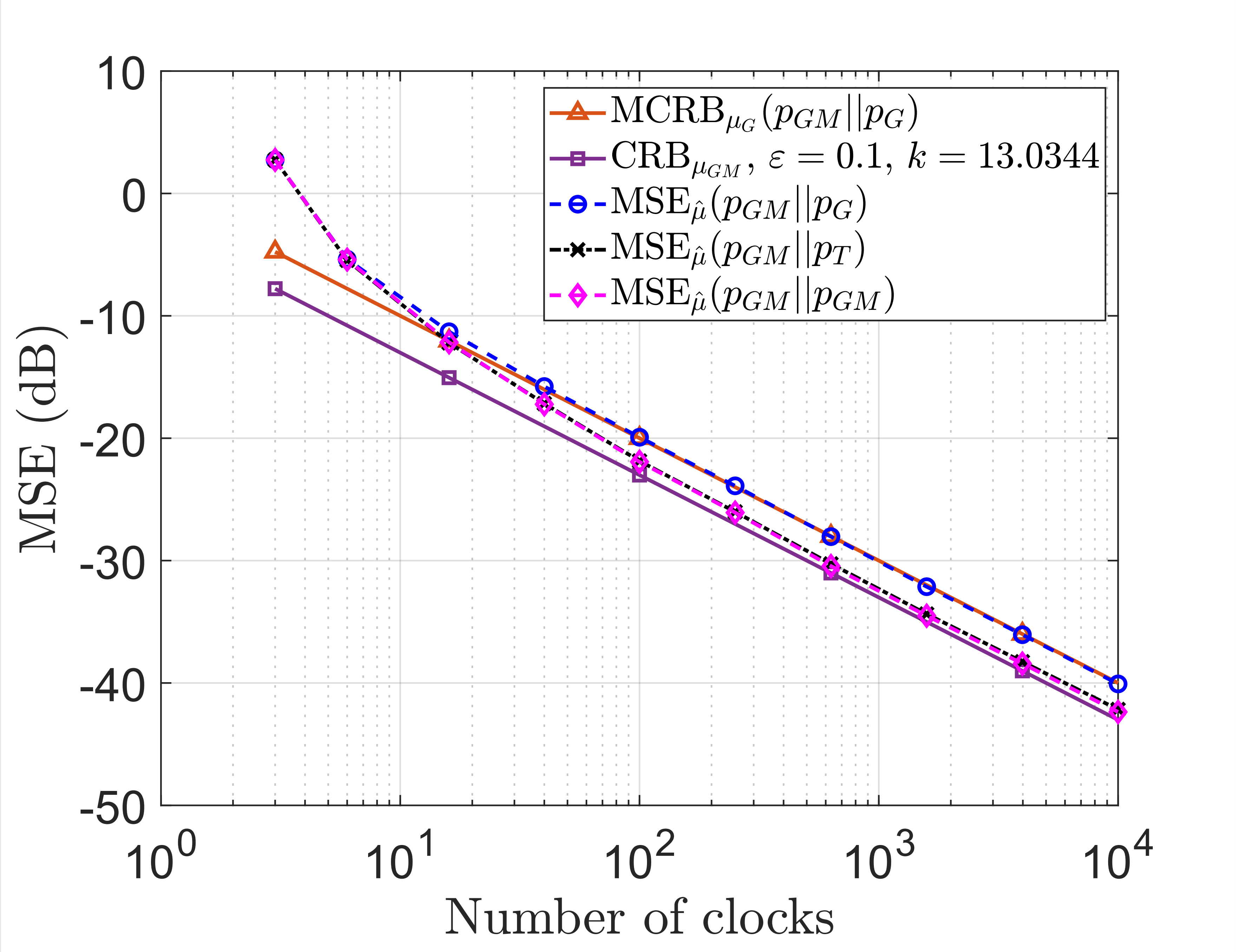
- Presented at EUSIPCO 2024 [3]

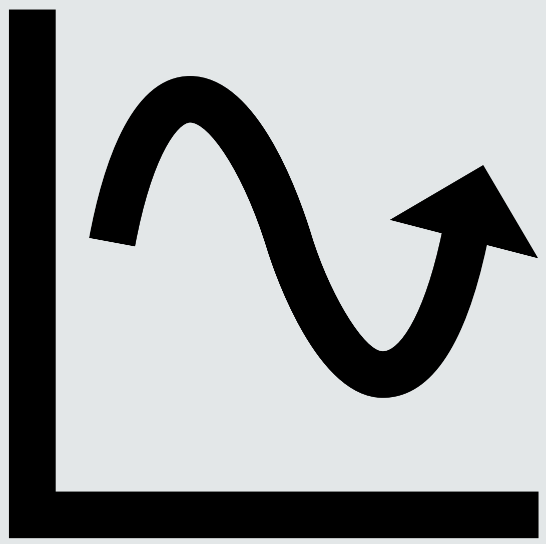
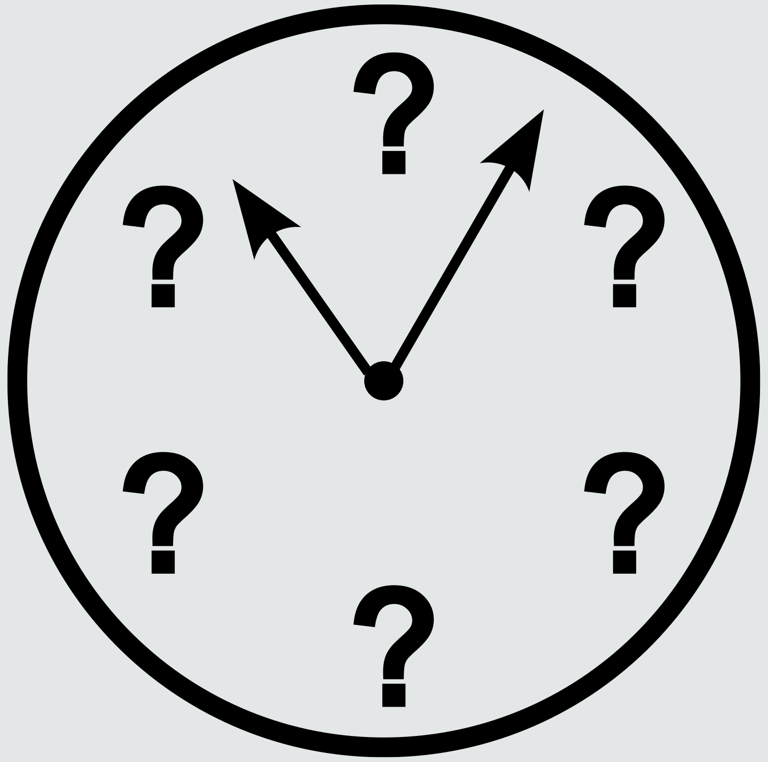
[2] McPhee, Hamish, et al. 'Misspecified Cramér-Rao Bounds for Anomalous Clock Data in Satellite Constellations'. *Proceedings of the 32nd Annual European Signal Processing Conference (EUSIPCO)*, Lyon, France, 2024.



MCRB – Bimodal Gaussian Mixture

- Yet to be published



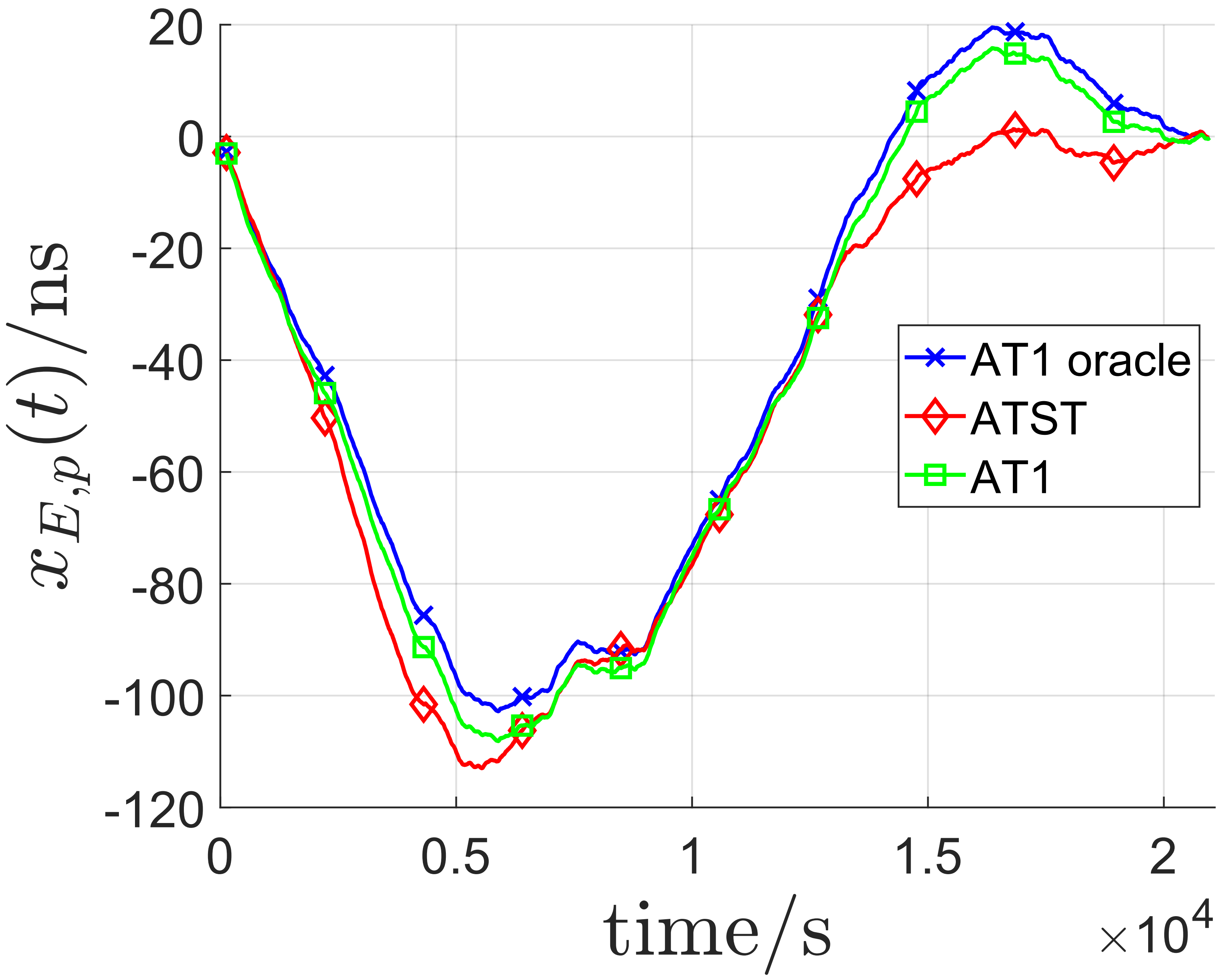

How does the ATST perform?

Simulation Setup

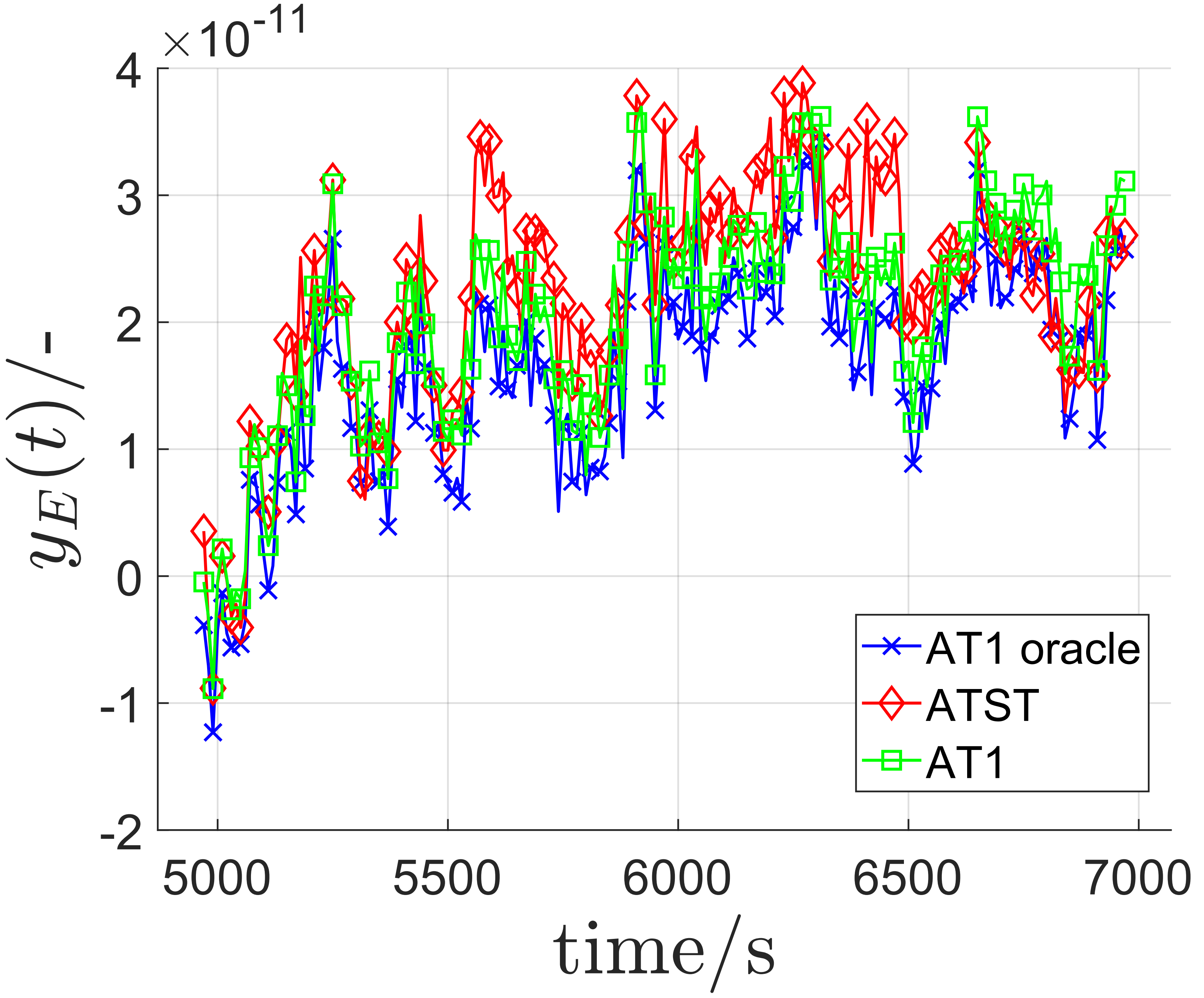
- Swarm of 50 satellites
- Homogeneous clock types
- 6 hours of simulation
- 10 second minimum interval
- Anomalies occur once on each clock

- Baseline AT1 oracle
  - Perfect detection of anomalies
  - Weights forced to zero
  - Recomputation of $x_{i,E}(t)$
- Performance Metrics
  - Phase continuity in $h_E(t) = x_{i,E}(t) - x_{i,p}(t)$
  - Frequency continuity
  - OADEV
  - Weights

Nominal

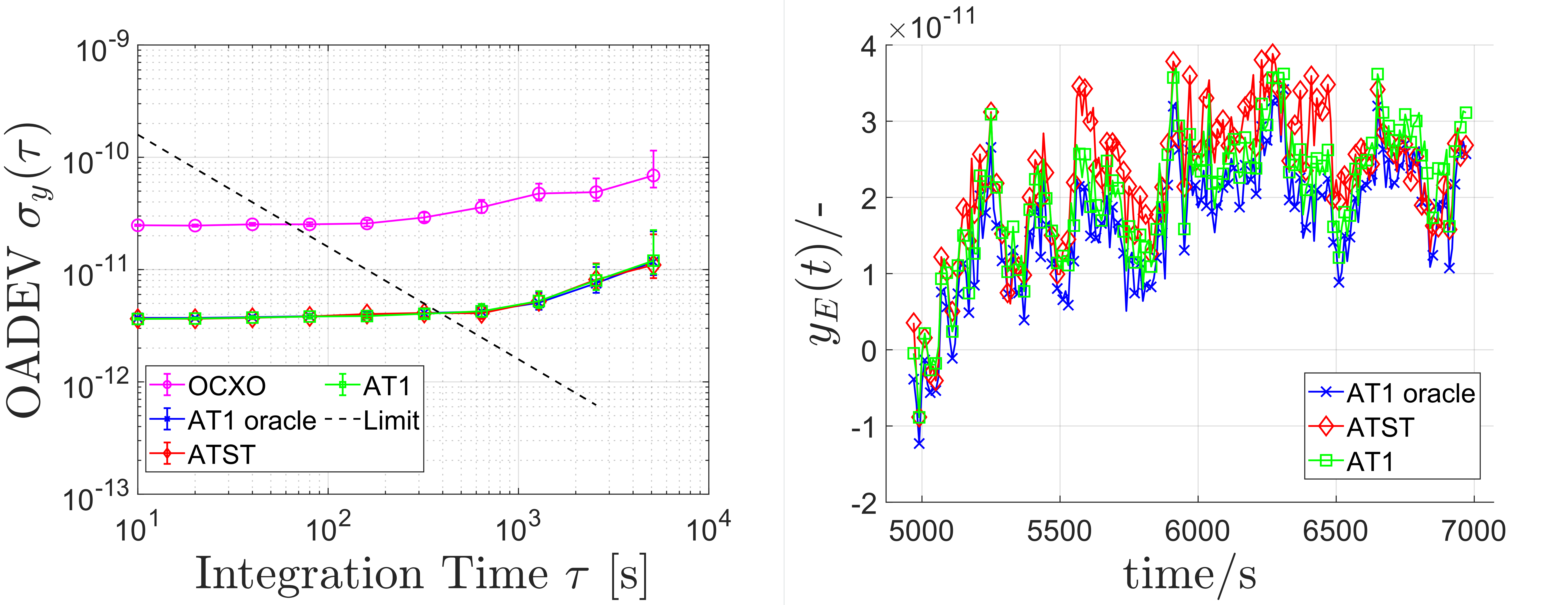


$x_{E,p}(t)/\mathrm{ns}$
20
0
-20
-40
-60
-80
-100
-120
0
0.5
1
1.5
2
$\times 10^4$
time/s
AT1 oracle
ATST
AT1



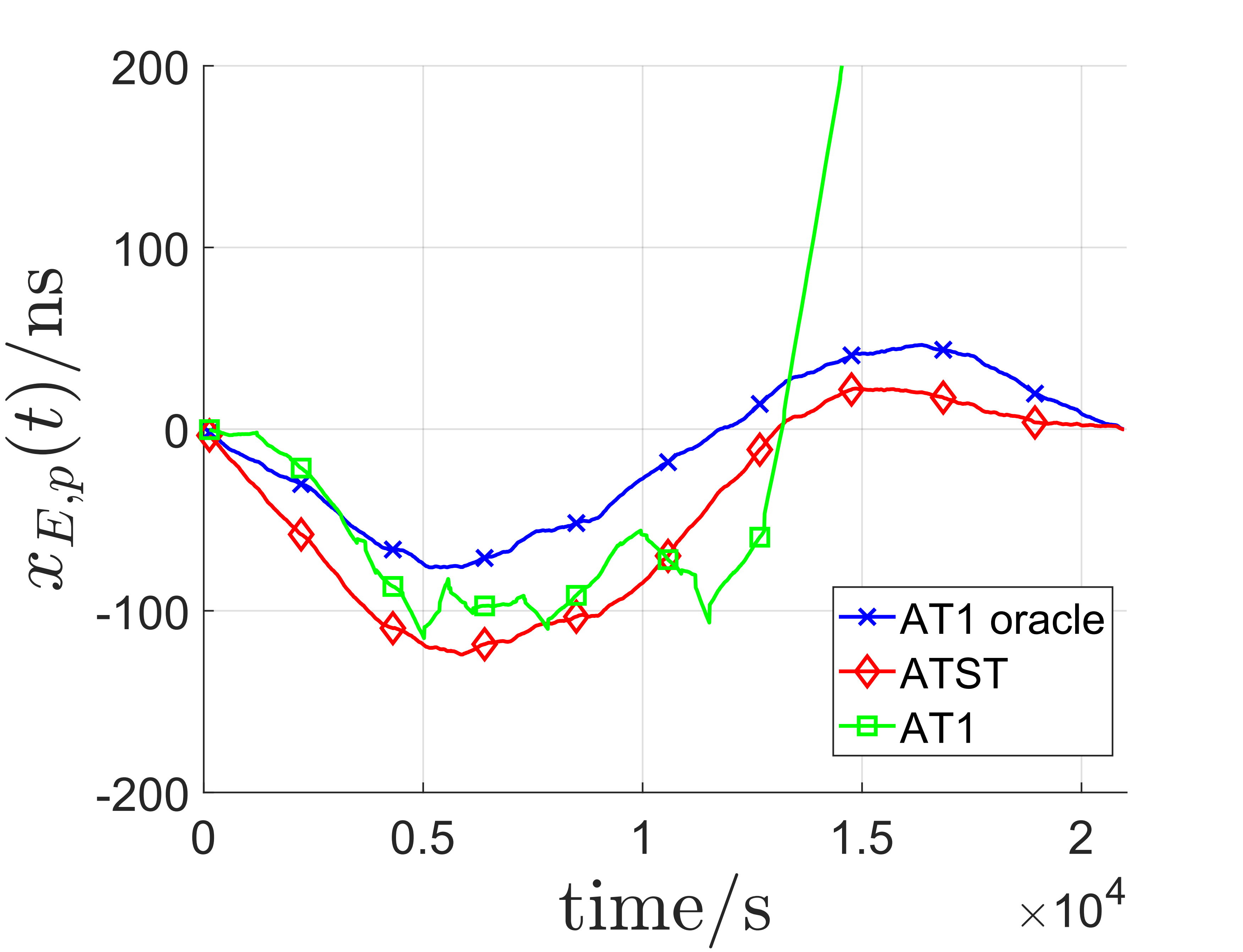
$\times 10^{-11}$
$y_E(t)/-$
4
3
2
1
0
-1
-2
5000
5500
6000
6500
7000
time/s
AT1 oracle
ATST
AT1

Nominal

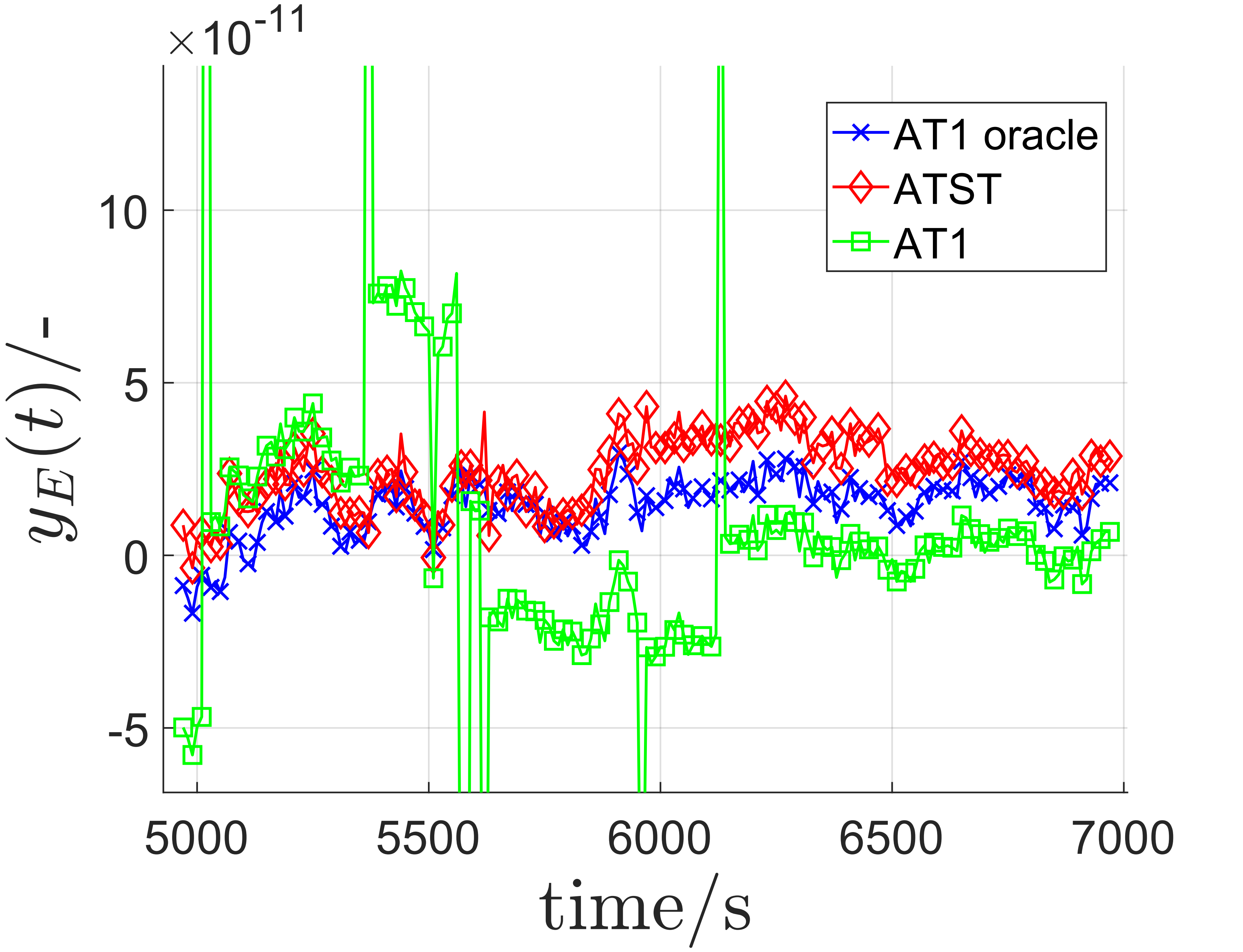


OADEV $\sigma_y(\tau)$
Integration Time τ [s]
OCXO
AT1
AT1 oracle
Limit
ATST
$y_E(t)/-$
time/s
$\times 10^{-11}$
AT1 oracle
ATST
AT1

Phase jumps

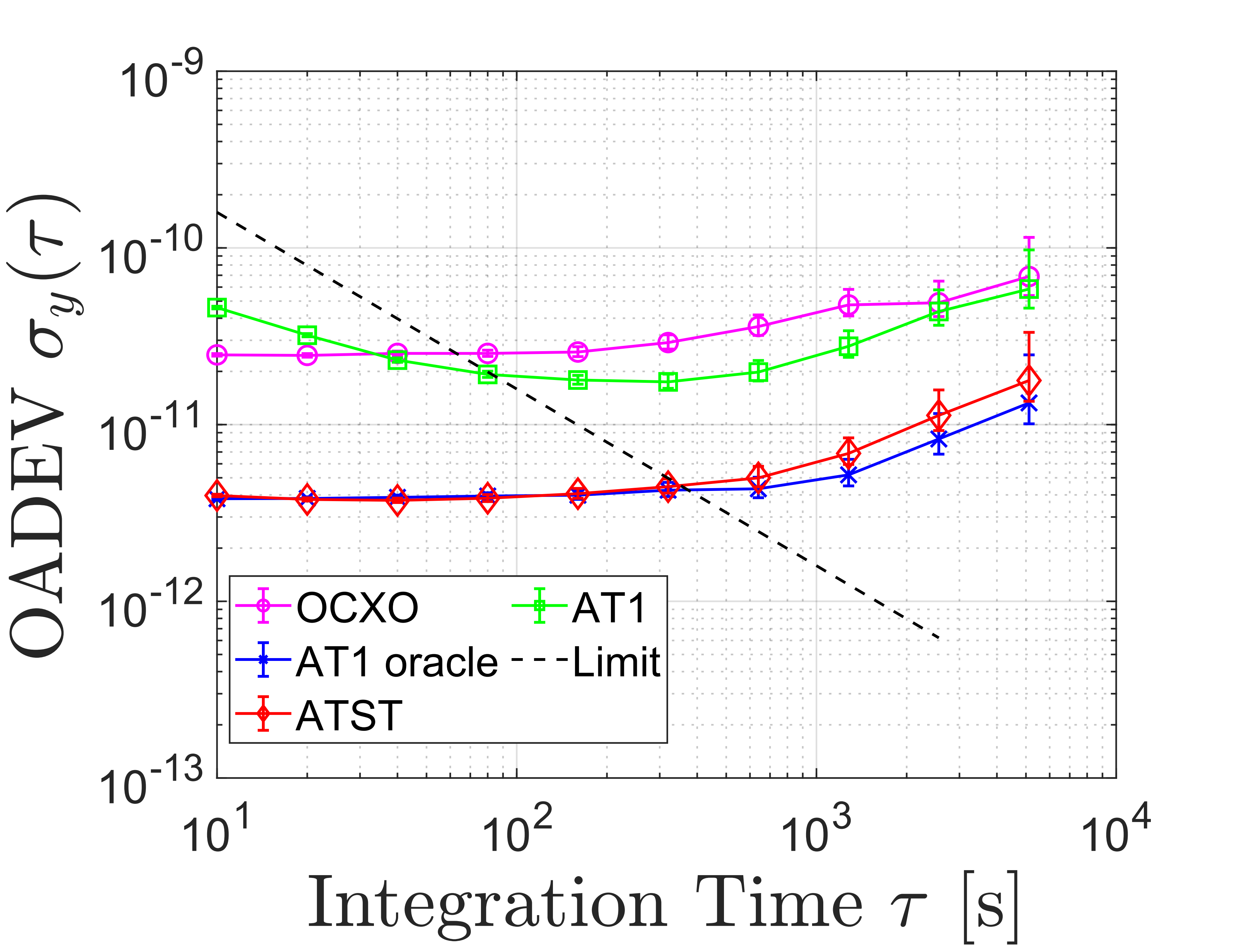


200
100
0
-100
-200
$x_{E,p}(t)/\mathrm{ns}$
0
0.5
1
1.5
2
$\times 10^4$
time/s
AT1 oracle
ATST
AT1

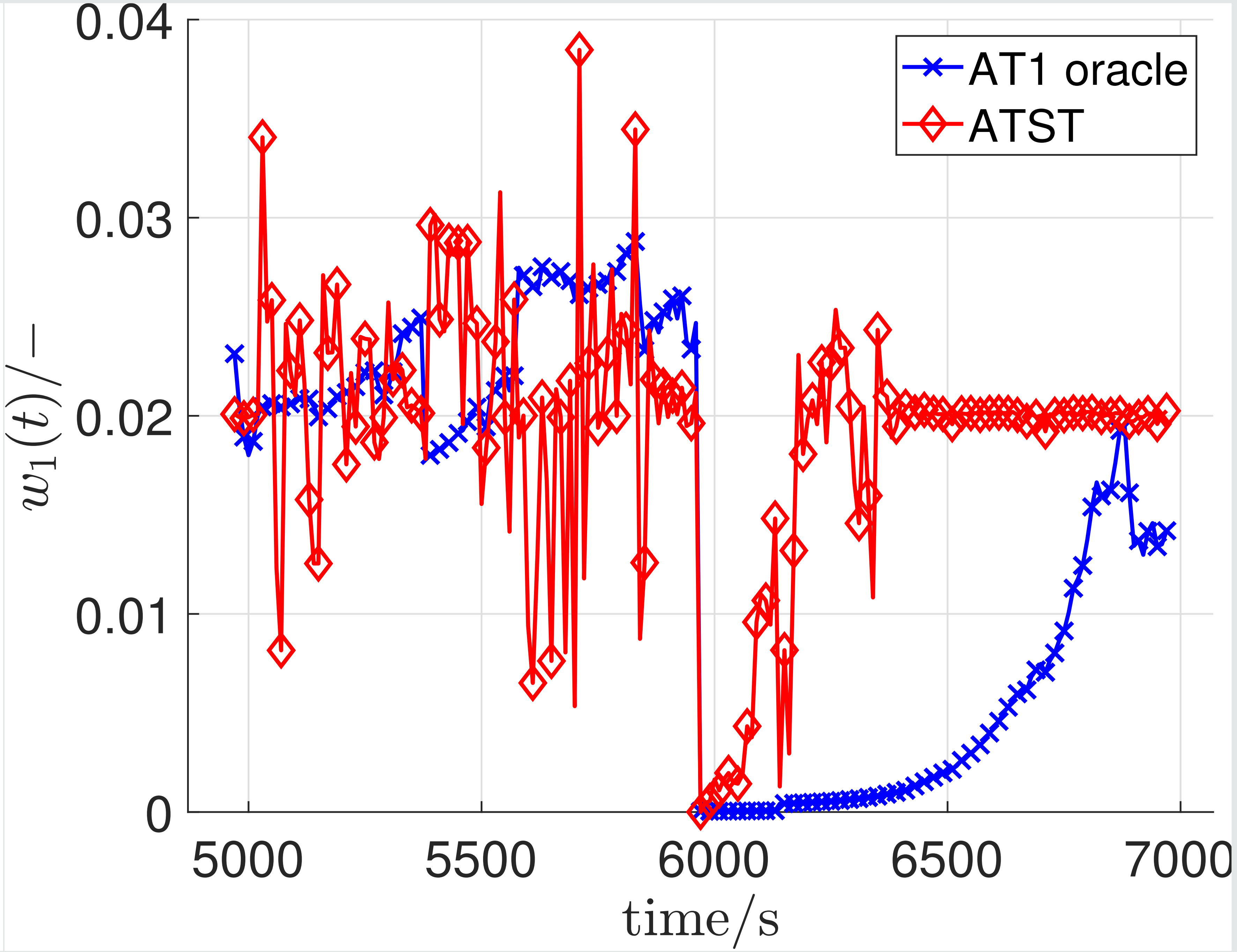


$\times 10^{-11}$
10
5
0
-5
$y_E(t)/-$
5000
5500
6000
6500
7000
time/s
AT1 oracle
ATST
AT1

Phase jumps

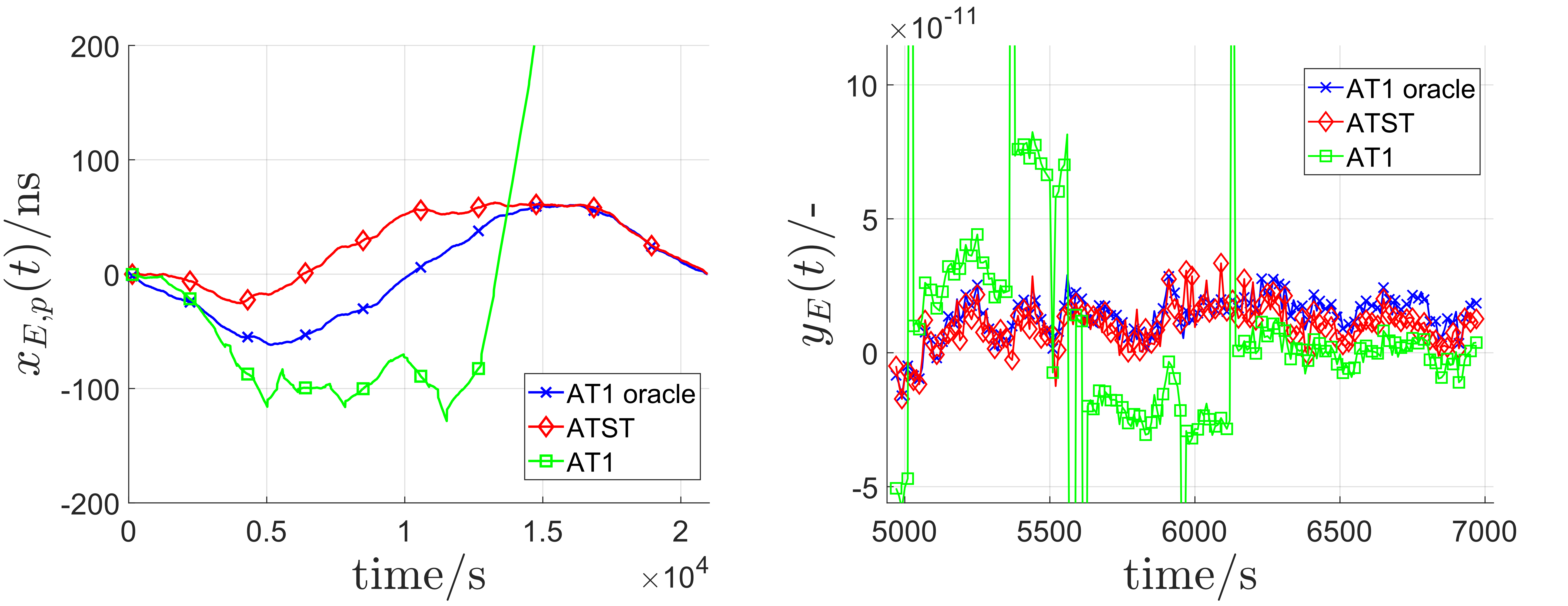


OADEV $\sigma_y(\tau)$
Integration Time τ [s]
OCXO
AT1
AT1 oracle
Limit
ATST

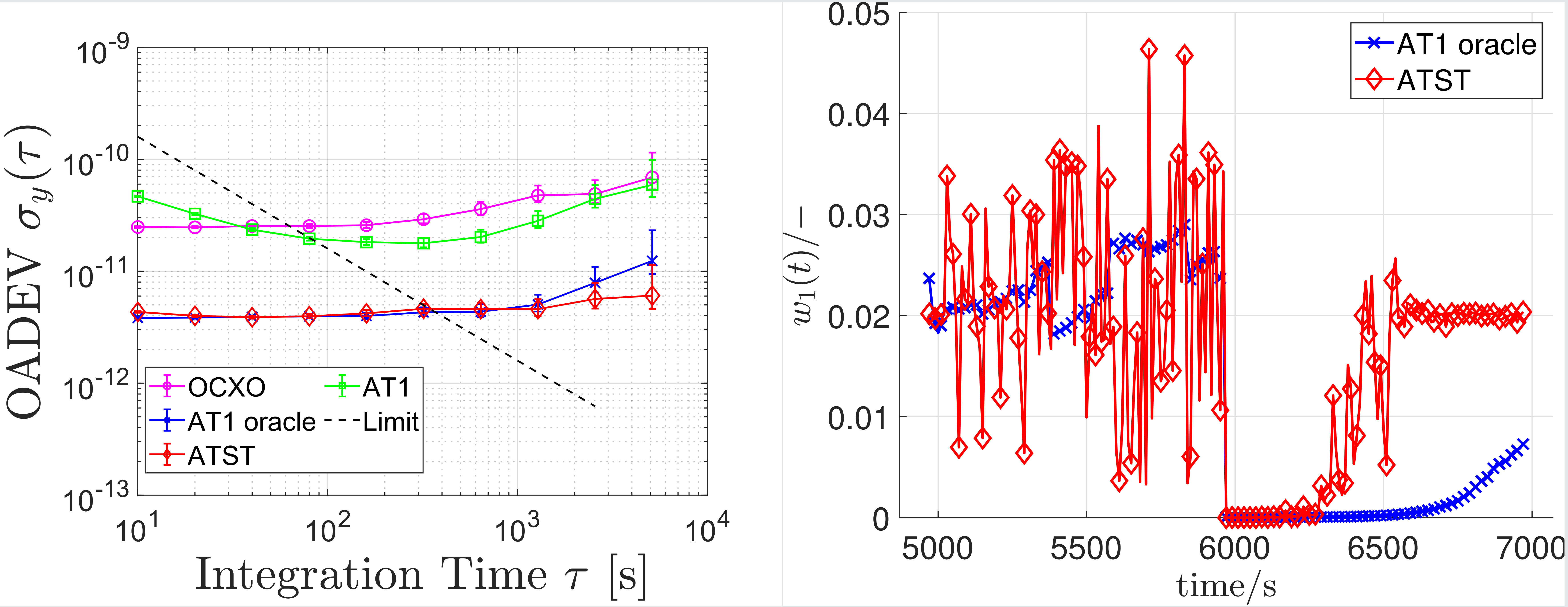


$w_1(t)/-$
time/s
AT1 oracle
ATST

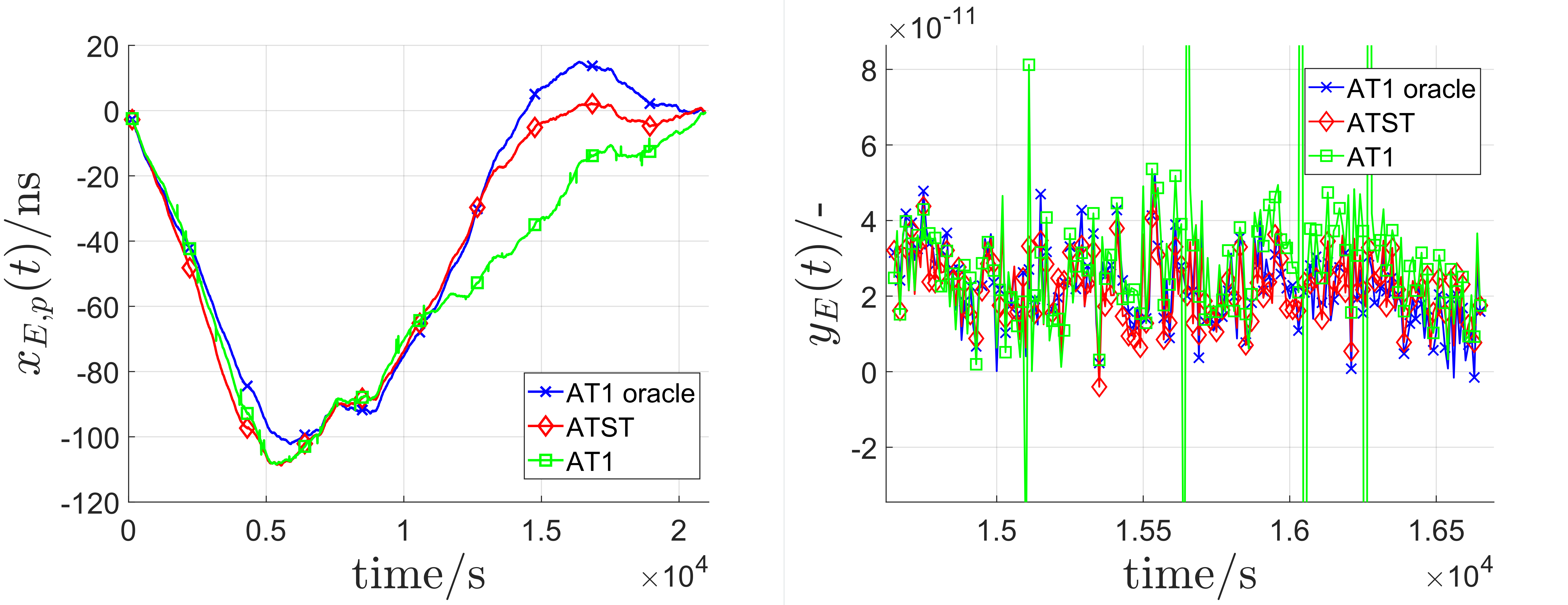
Frequency jumps



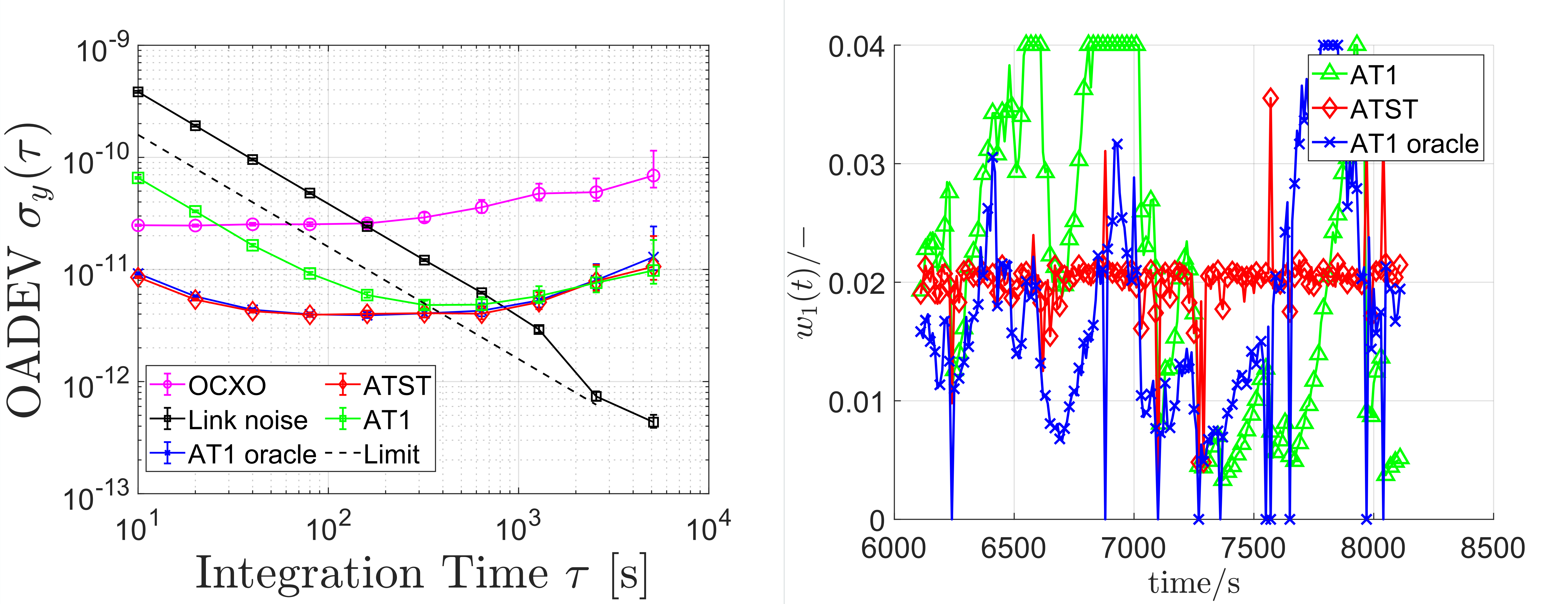
Frequency jumps



Measurement anomalies



Measurement anomalies



A new method to tell the time?

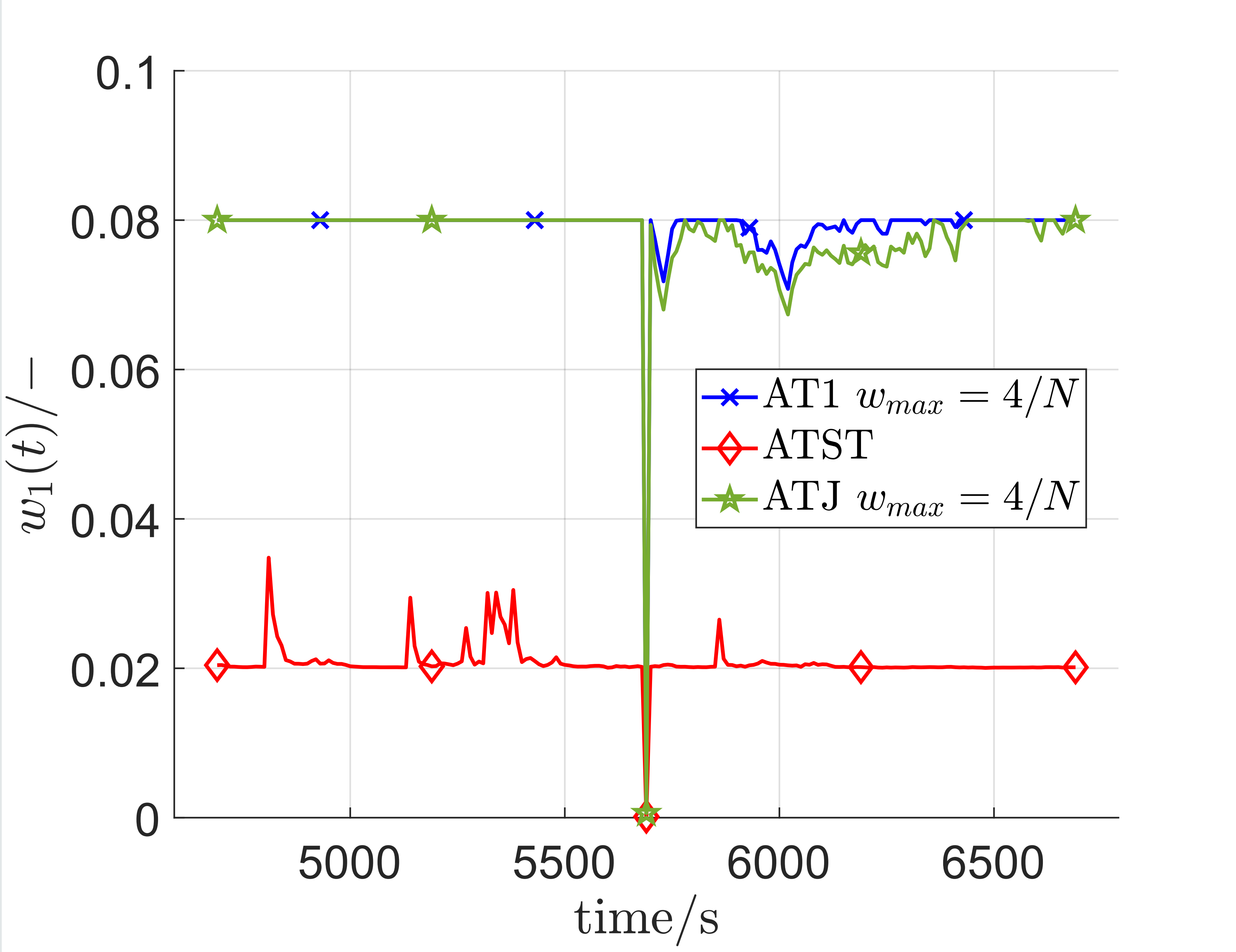
Conclusions

- ATST Robust time scale algorithm
  - Same weighting method for many anomalies
  - Restricted by number and type of clocks

- MCRB to assess performance of robust estimation
  - Defines required number of clocks
  - Equivalence of Heavy-tailed distributions
  - Not universal

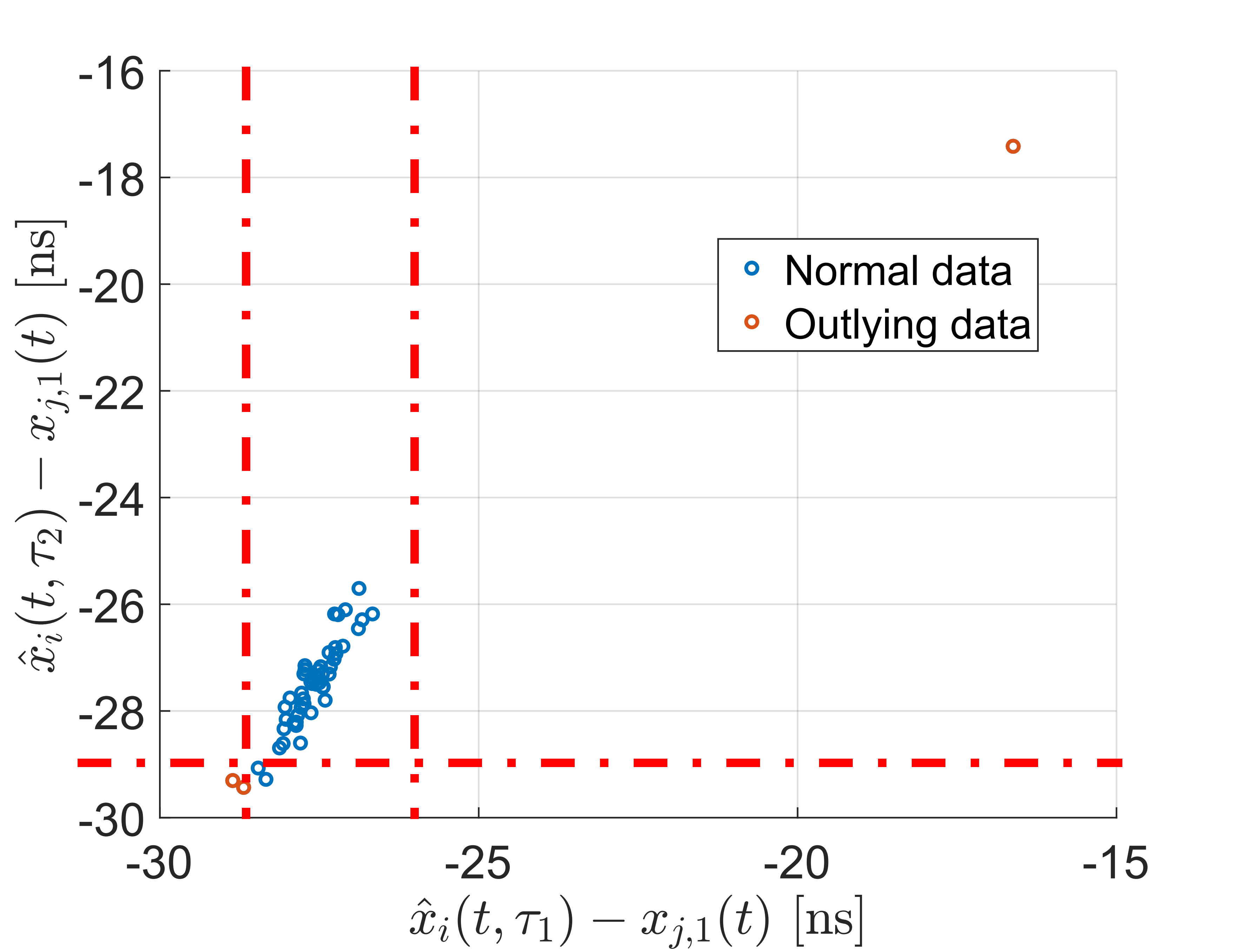
Future Work

- Combine AT1 and ATST



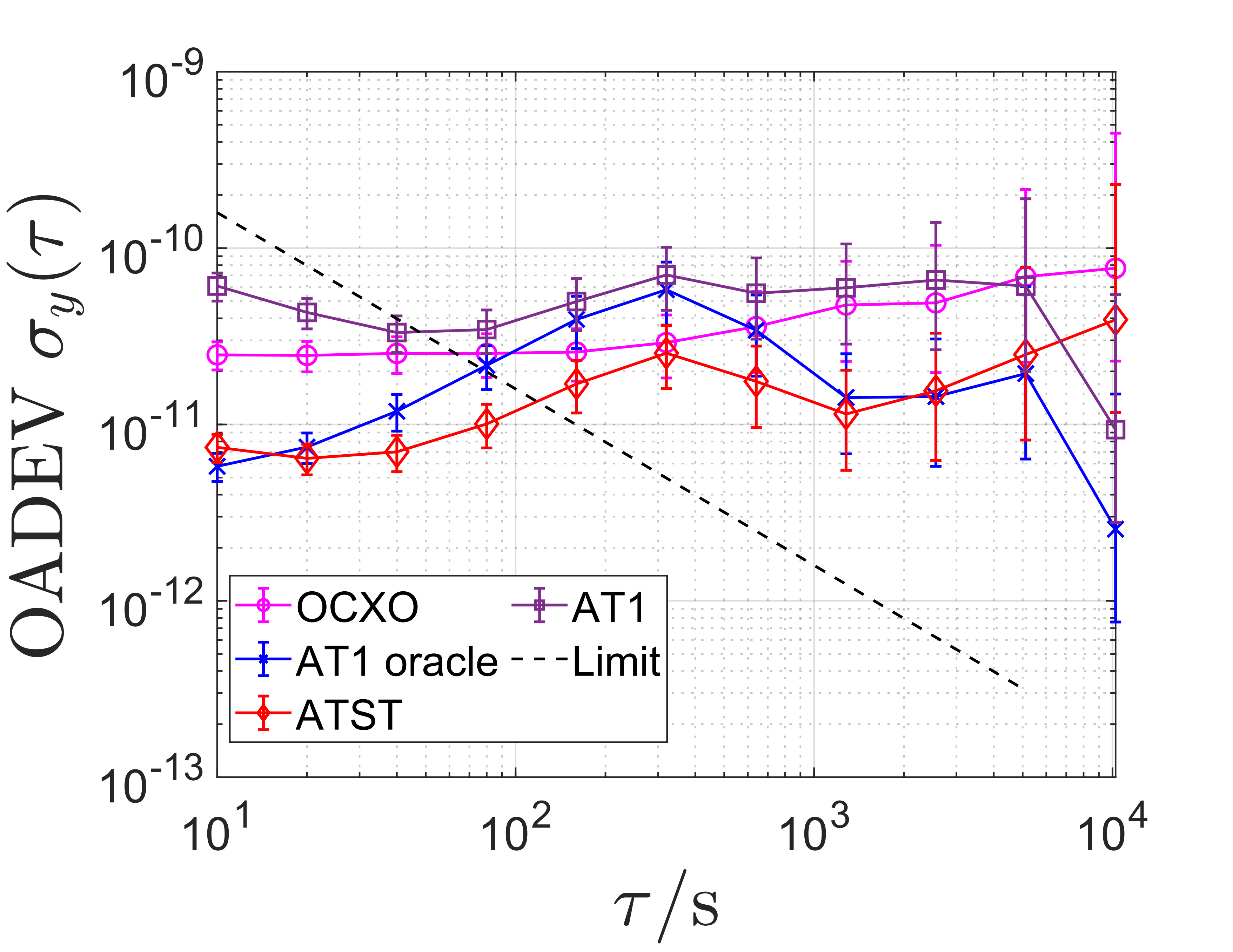
Future Work

- Combine AT1 and ATST
- Machine Learning



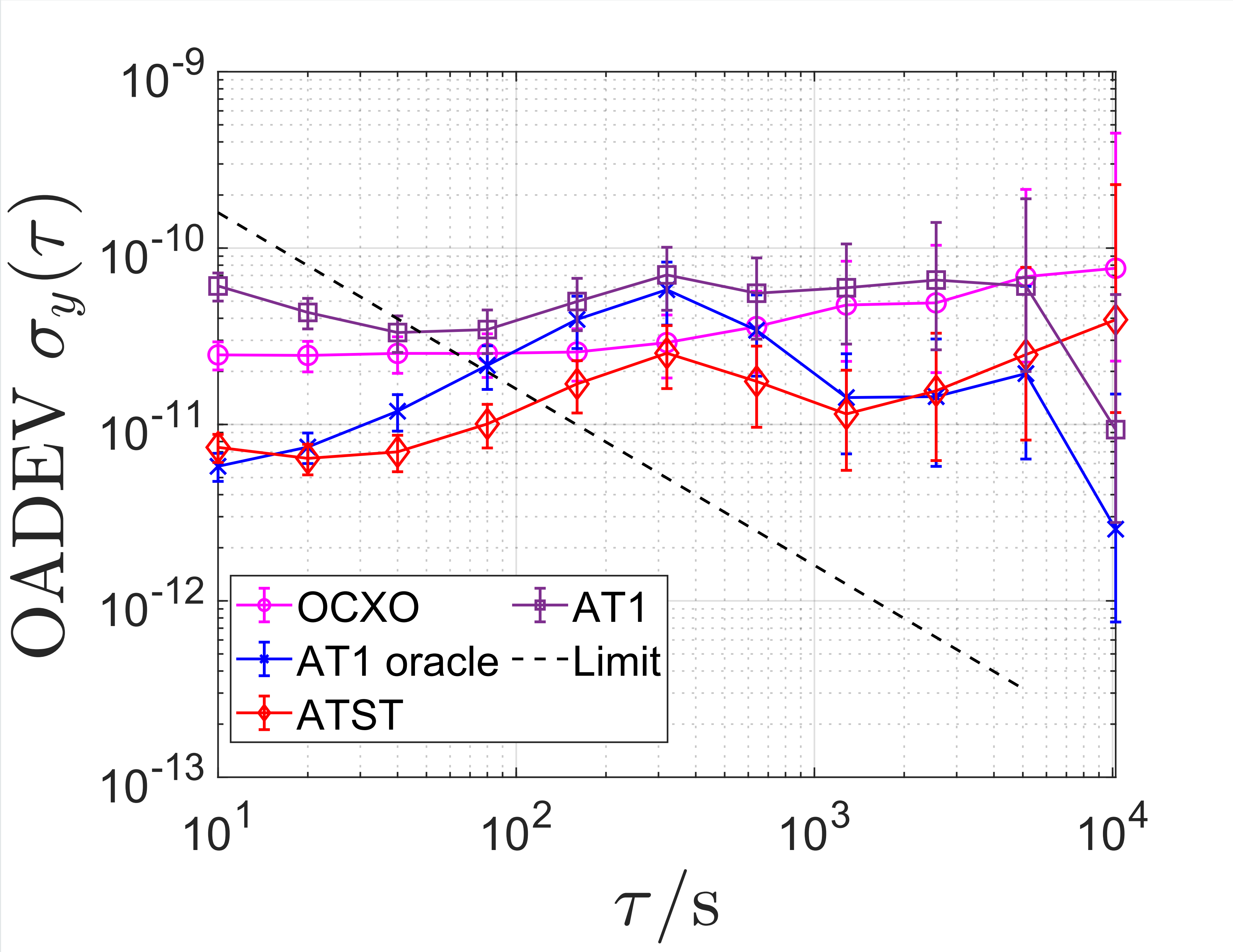
Future Work

- Combine AT1 and ATST
- Machine Learning
- Transient Anomalies

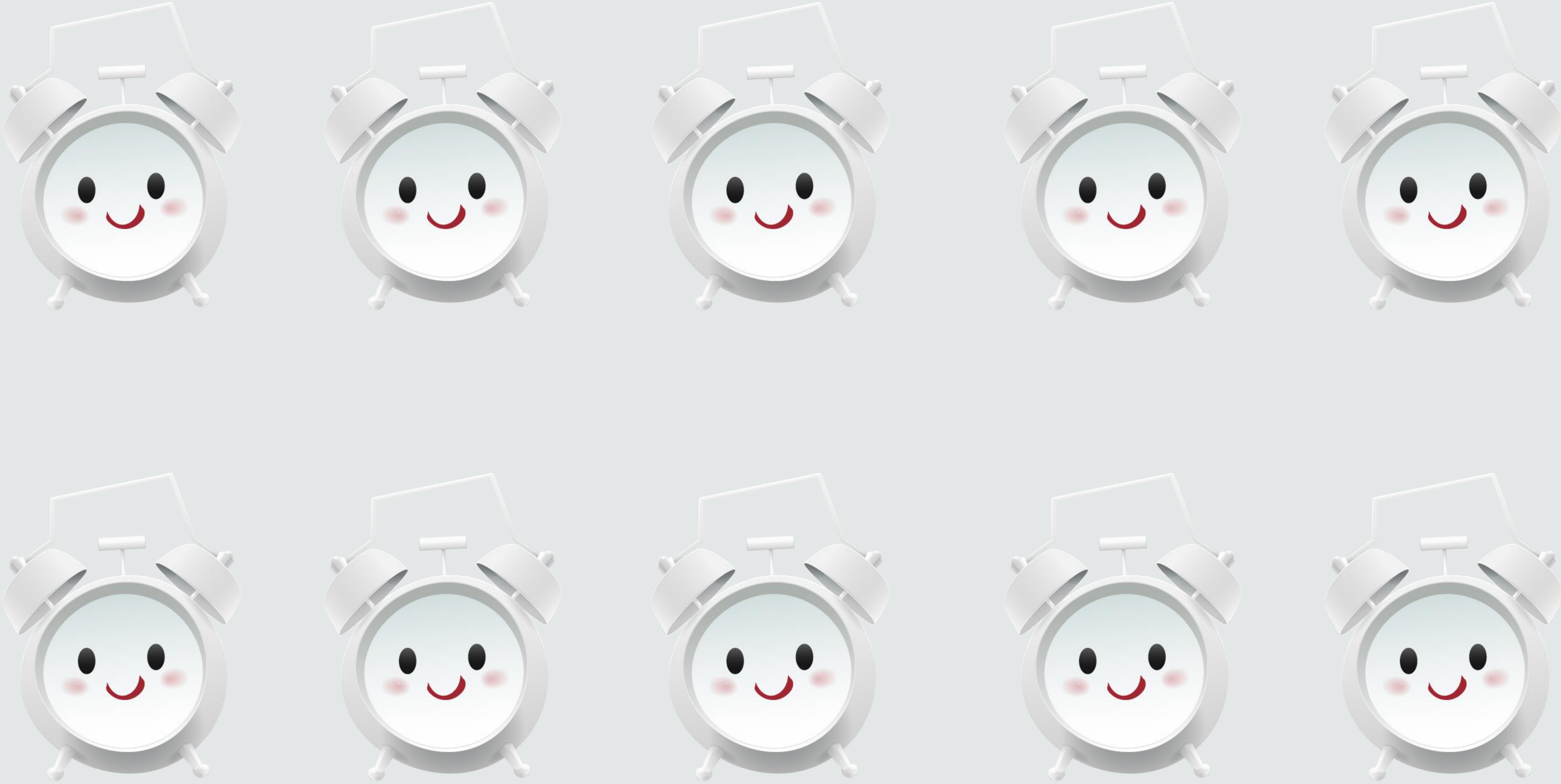


Future Work

- Combine AT1 and ATST
- Machine Learning
- Transient Anomalies
- Real data



Collecting Real Data



Real Data

Thank you for your time!

List of Publications

1. McPhee, Hamish, et al. ‘Exploiting Redundant Measurements for Time Scale Generation in a Swarm of Nanosatellites’. *Proceedings of the 37th Annual European Frequency and Time Forum (EFTF)*, Neuchâtel, Switzerland, 2024.

2. McPhee, Hamish, et al. ‘A Robust Time Scale for Space Applications Using the Student’s t-Distribution’. *Metrologia*, vol. 61, no. 5, Sept. 2024, p. 055010.

3. McPhee, Hamish, et al. ‘Misspecified Cramér-Rao Bounds for Anomalous Clock Data in Satellite Constellations’. *Proceedings of the 32nd Annual European Signal Processing Conference (EUSIPCO)*, Lyon, France, 2024.

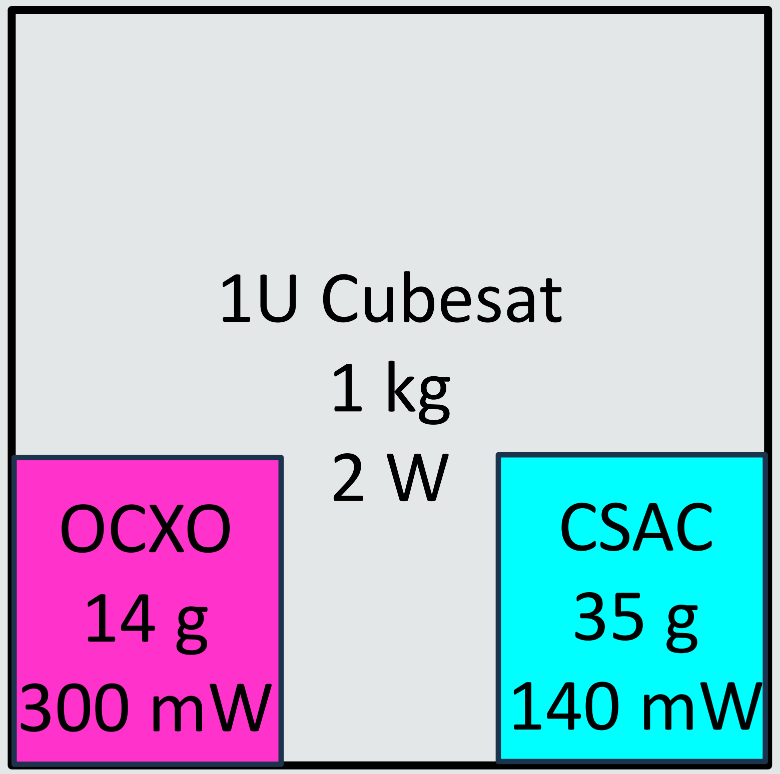
4. McPhee, Hamish, et al. ‘A Robust Time Scale Based on Maximum Likelihood Estimation’. *Proceedings of the 54th Annual Precise Time and Time Interval Systems and Applications Meeting*, Institute of Navigation (IoN), Long Beach, California, 2023.

Backup Slides

Clocks in space

- Important constraints for nanosatellites:
  - –Size (volume)
  - –Weight
  - –Power
- Mission requirements:
  - –Frequency Stability
  - –Continuity

- Oven Controlled Crystal Oscillator (OCXO)
- Chip Scale Atomic Clock (CSAC)



Clock noises

Noise Types	α	AVAR
White Phase	2	$\frac{3f_H}{4\pi^2}\frac{h_2}{\tau^2}$
Flicker Phase	1	$\frac{1.731 - \log(2) + 3\log(2\pi f_H \tau)}{4\pi^2}\frac{h_1}{\tau^2}$
White Frequency	0	$\frac{1}{2}\frac{h_0}{\tau}$
Flicker Frequency	-1	$2\log(2) h_{-1}$
Random Walk Frequency	-2	$\frac{2\pi^2}{3} h_{-2}\tau$

Clock Simulation

From AVAR, get h_α then generate random noise according to associated variances $Q_d(\beta)$

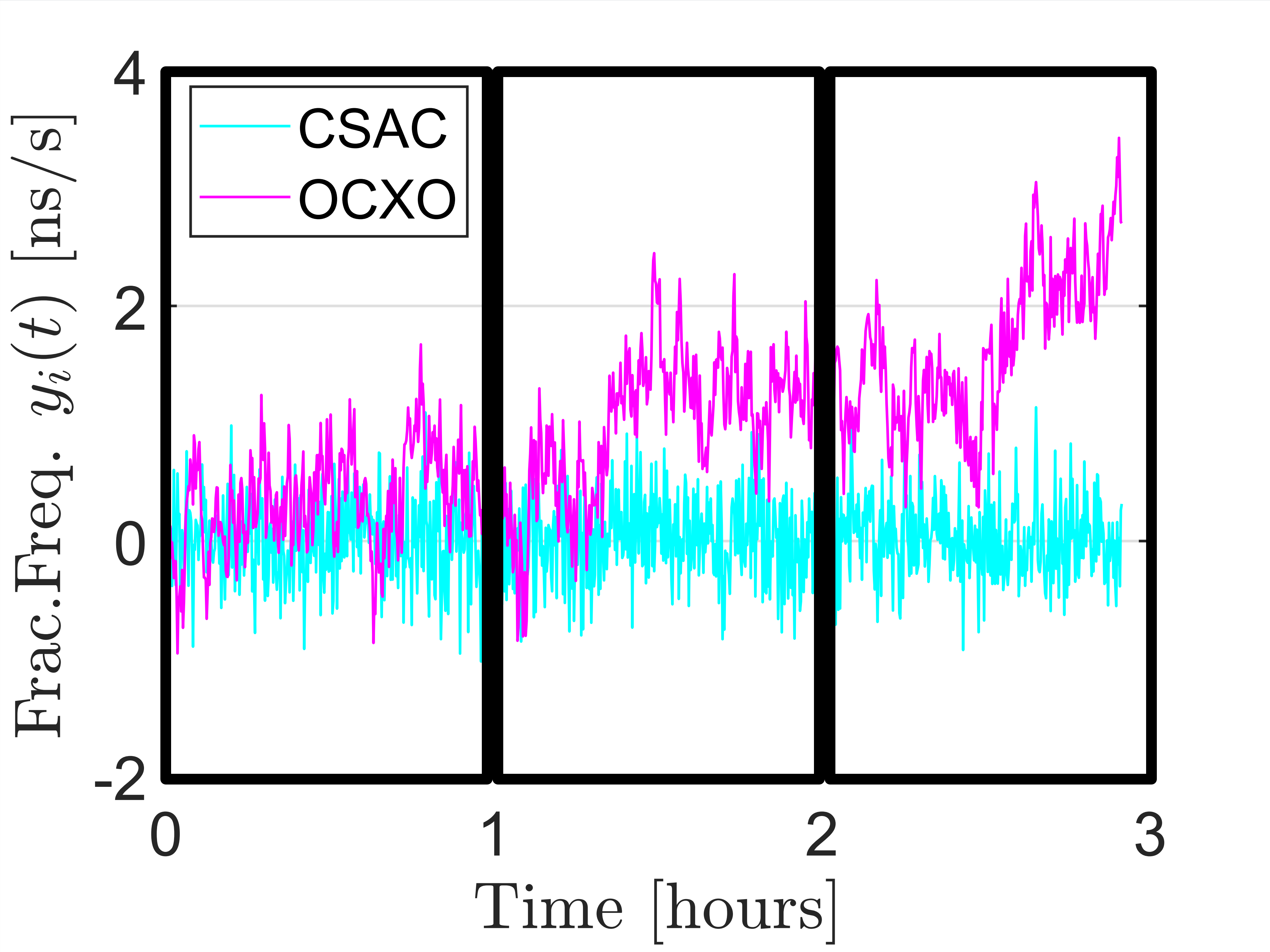
$$S_y(f) = \sum_{\alpha=-2}^{2} h_\alpha f^\alpha, \qquad S_x(f) = \frac{S_y(f)}{(2\pi f)^2} = \sum_{\beta=-4}^{0} g_\beta f^\beta,$$

$$Q_d(\beta) = \frac{g_\beta}{2(2\pi)^\beta \tau_0^{\beta+1}}$$

Note: $Q_d(\beta)$ is multiplied by a randomly generated real number to simulate uniqueness of independent clock behaviours

Calculating OADEV

$$\sigma_y(m\tau_0) = \sqrt{\frac{1}{L}\sum_{j=1}^{L/2m^2}\sum_{k=j}^{j+m-1}(\bar{y}_{k+m}-\bar{y}_k)^2}$$



Weights

- Kalman Filter (GPS Composite clock)

$$\boldsymbol{x}(t) = \boldsymbol{K}(\boldsymbol{z}(t) - \boldsymbol{H}\widehat{\boldsymbol{x}}(t))$$

$$\begin{bmatrix} x_{1,E}(t) \\ \vdots \\ x_{N,E}(t) \end{bmatrix} = \begin{bmatrix} K_{1,1} & \cdots & K_{1,N-1} \\ \vdots & \ddots & \vdots \\ K_{N-1,1} & \cdots & K_{N-1,N-1} \end{bmatrix} \left(\begin{bmatrix} z_{1,1}(t) \\ \vdots \\ z_{N,1}(t) \end{bmatrix} - \left(\begin{bmatrix} \hat{x}_{1,E}(t) - \hat{x}_{1,E}(t) \\ \vdots \\ \hat{x}_{N,E}(t) - \hat{x}_{1,E}(t) \end{bmatrix} \right) \right)$$

$$x_{i,E}(t) = \sum_{j=1}^{N} w_j(?) \left(\hat{x}_{j,E}(t) - z_{j,i}(t) \right)$$

$$\boldsymbol{K} = \boldsymbol{P}_{k|k-1}\boldsymbol{H}^T \left(\boldsymbol{H}\boldsymbol{P}_{k|k-1}\boldsymbol{H}^T + \boldsymbol{R} \right)^{-1}$$

Weights

- Inverse of frequency variance (ALGOS), expanded in [PTTI 23]

$$w_i(t) = \frac{1}{\hat{\sigma}_{y_i}^2(t)}$$

$$\hat{\sigma}_{y_i}^2(t) = \frac{1}{L}\sum_{m=0}^{L-1} p_m \left(y_{i,s}(t - m\tau) - \bar{y}_i(t)\right)^2$$

$$\bar{y}_i(t) = \frac{1}{L}\sum_{m=0}^{L-1} p_m y_{i,s}(t - m\tau)$$

$$y_{i,s}(t) = \frac{x_{i,E}(t) - x_{i,E}(t - \tau)}{\tau}$$

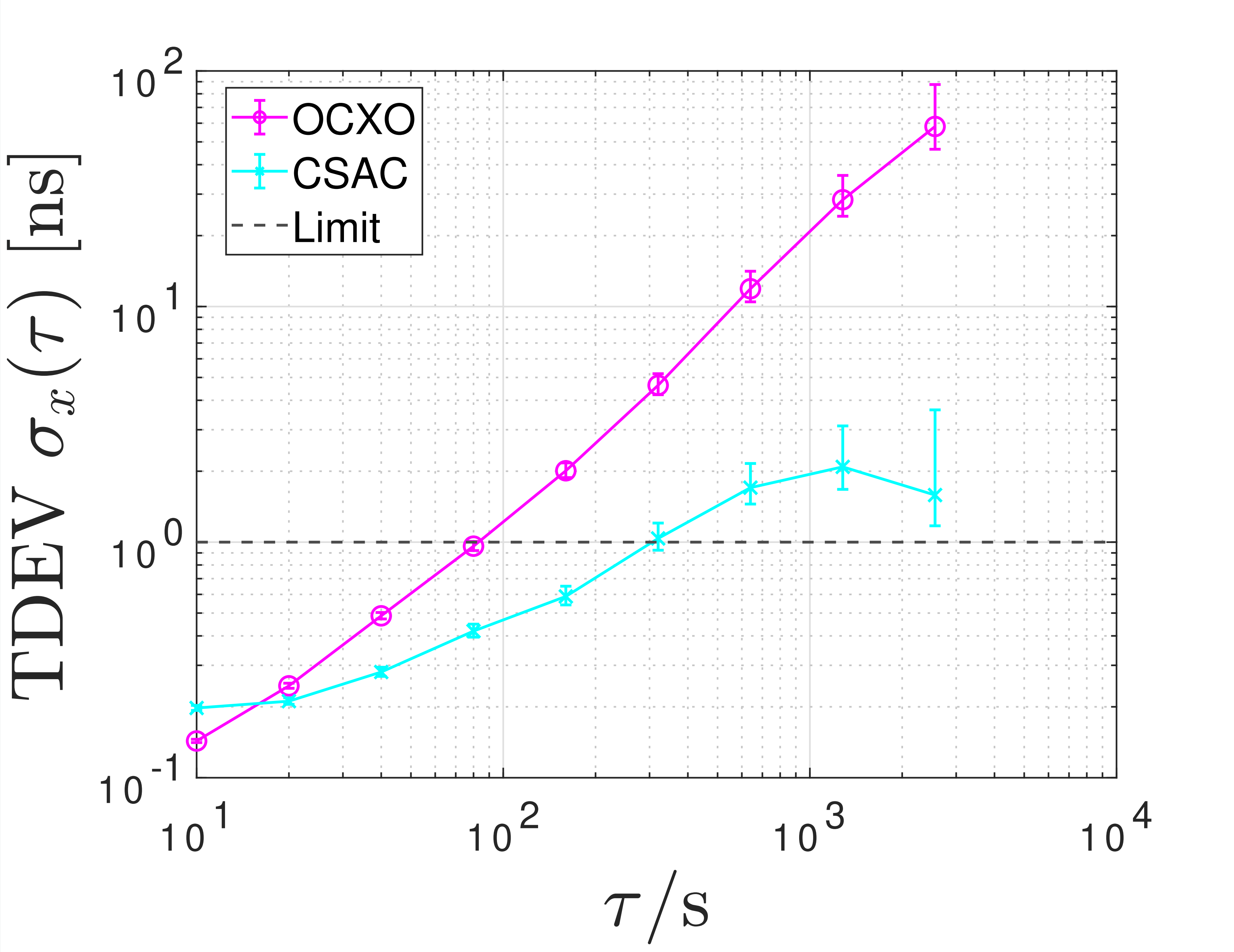
Time Accuracy

$$\sigma_x(\tau) = \frac{\tau}{\sqrt{3}} \sqrt{\frac{1}{L} \sum_{j=1}^{L/2m^4} \left\{ \sum_{i=j}^{j+m-1} \left(\sum_{k=i}^{i+m-1} (y_{k+m} - y_k) \right) \right\}^2}$$

- Time Deviation (TDEV)
- Max. for coherent networks [ITU]

$$\sigma_x(\tau) \leq 10 \text{ ns}$$

- Synchronization metric
- CSAC provides larger operational range
- Time scale improves range



Backup

Continuity Measure

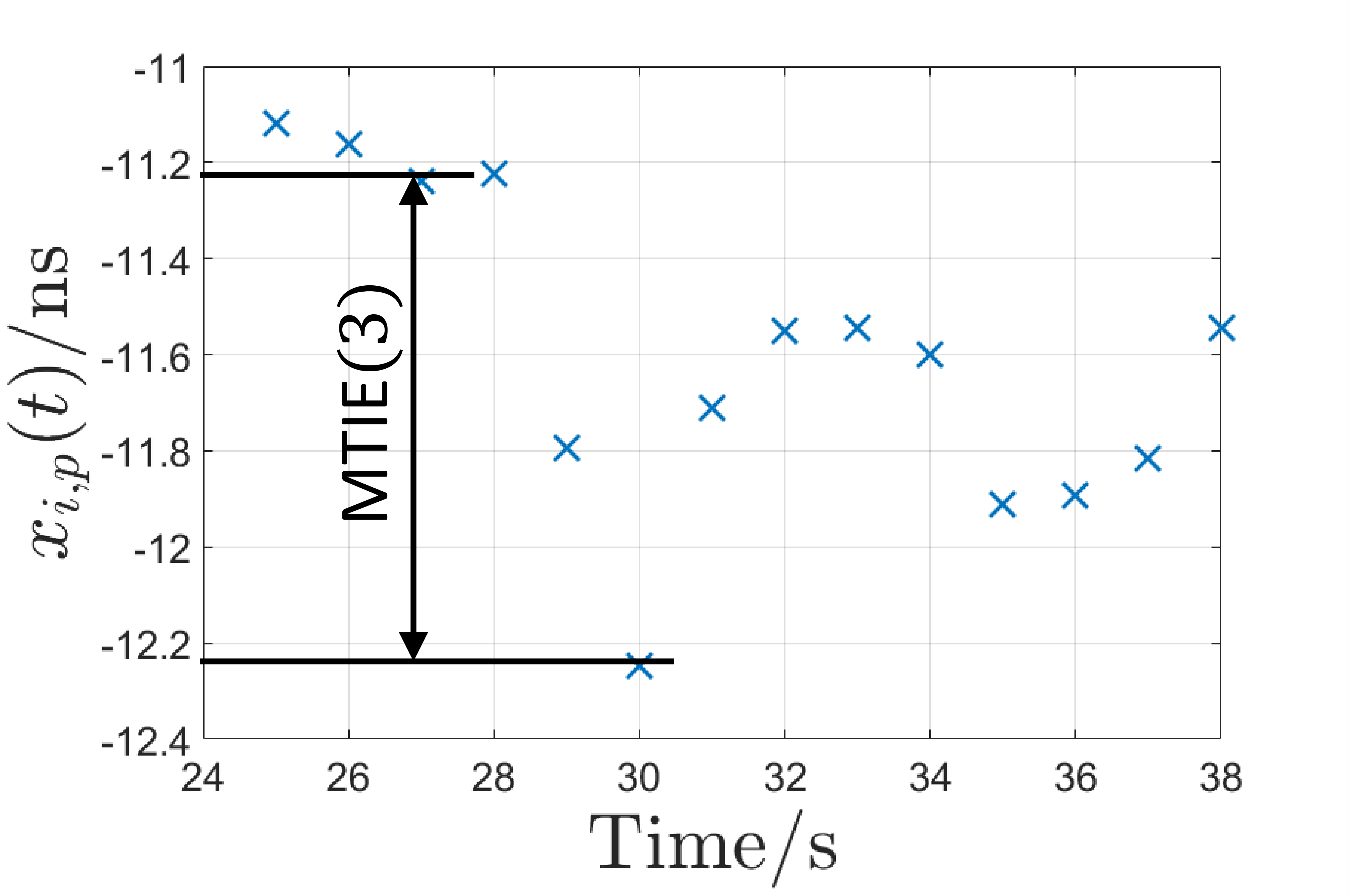
- Maximum Time Interval Error (MTIE)

$$\mathrm{MTIE}(\tau) = \max_{t>0}\{x_i(t) - x_i(t-\tau)\}$$

- Max. for coherent networks [ITU]

$$\mathrm{MTIE} \leq 30\ \mathrm{ns}$$

- Frequency continuity equally important
- CSAC provides larger operational range
- Time scale improves range



Backup

Continuity Measure

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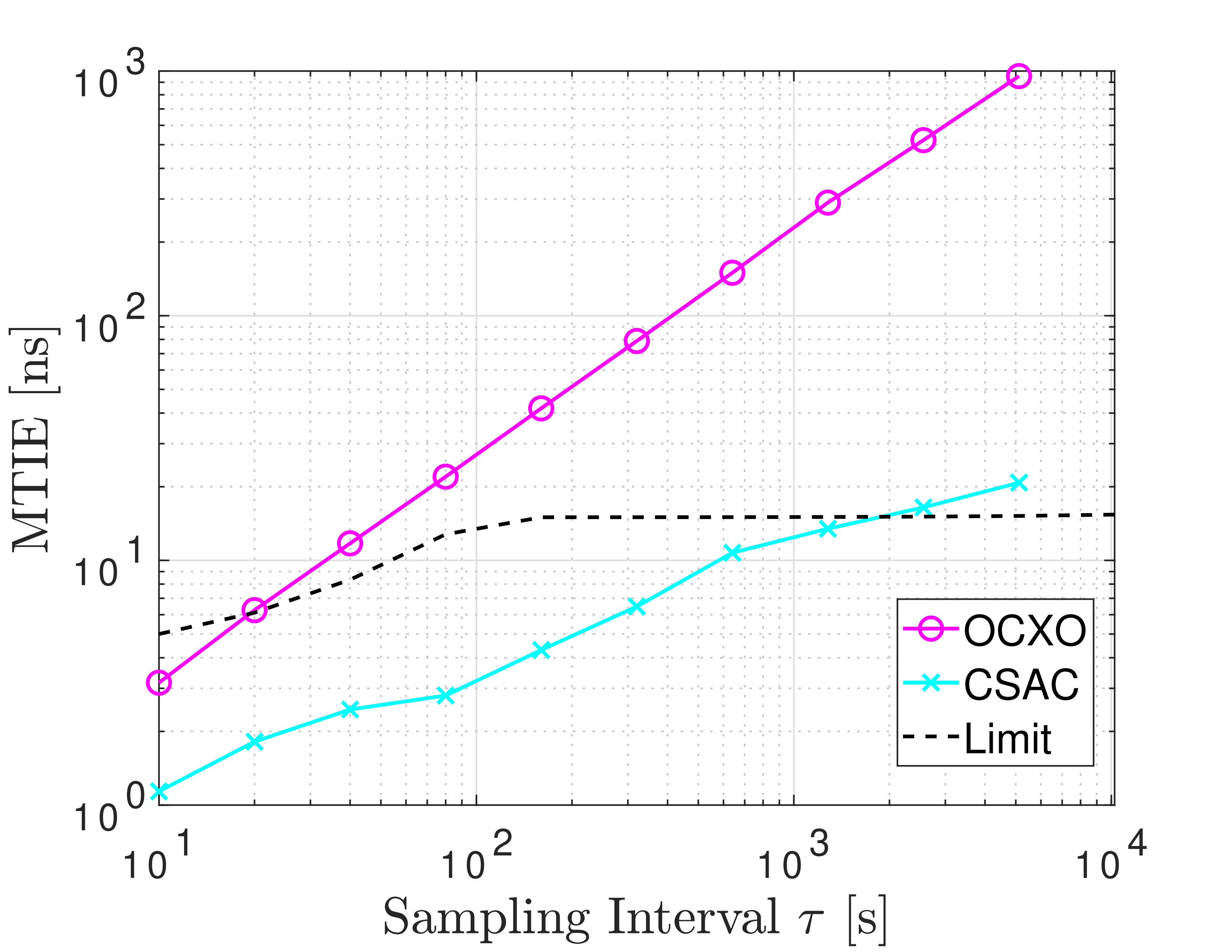
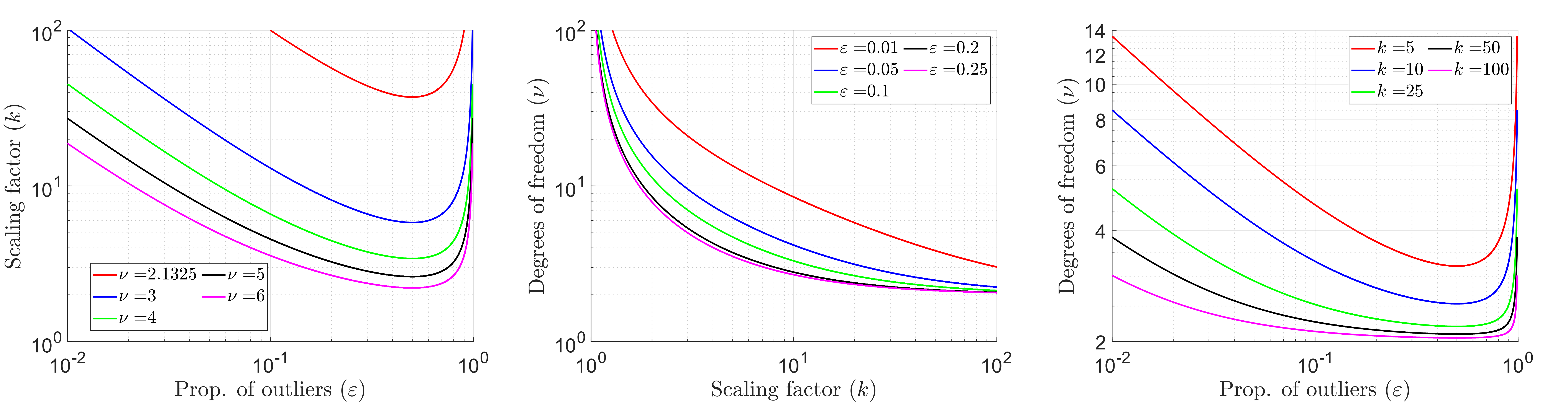


Equivalence of Heavy-tailed distributions

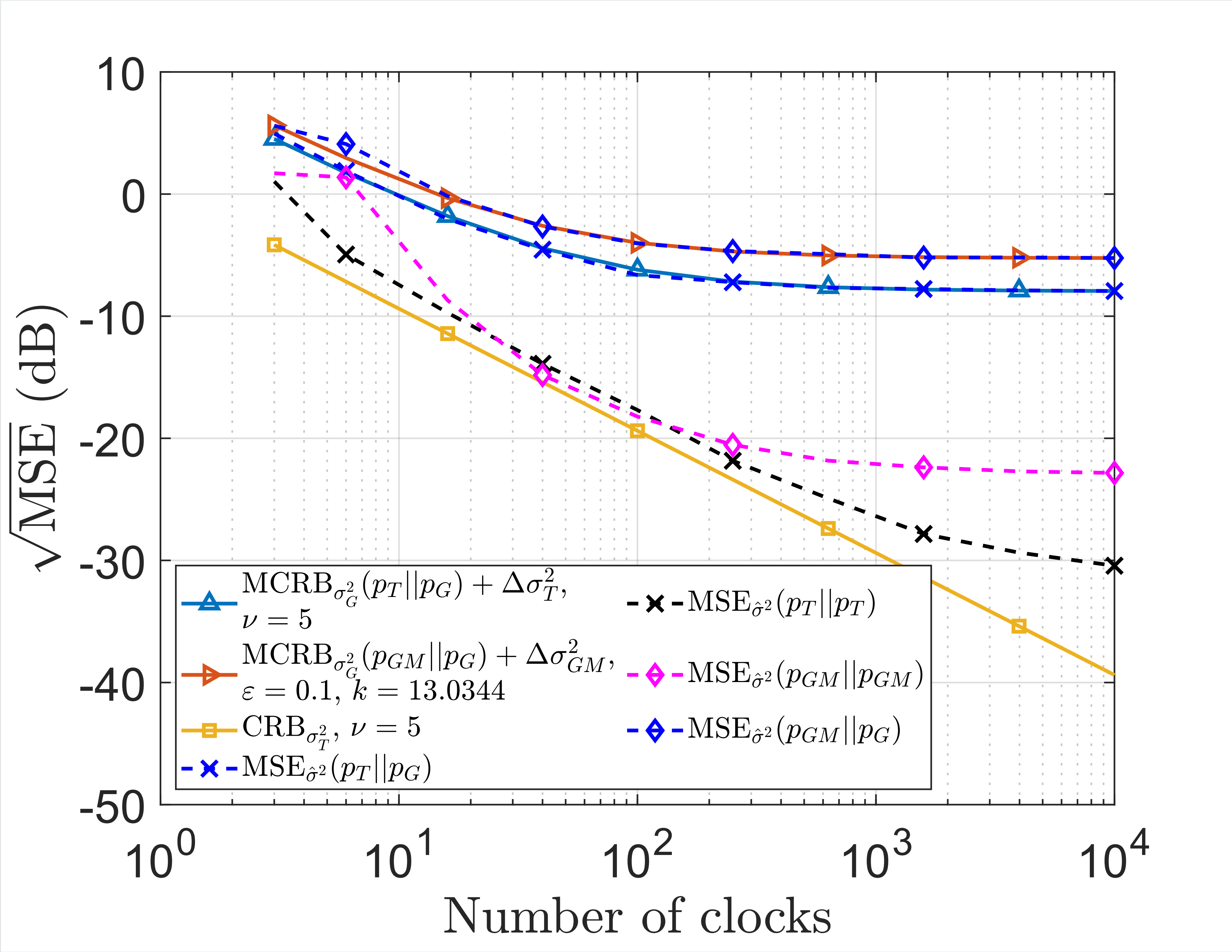
Equating CRBs:

$$\left(\frac{\nu+3}{\nu+1}\right)(\varepsilon(k-1)+1) = \left(\frac{\nu}{\nu-2}\right)\left(\frac{k}{k-\varepsilon(k-1)}\right)$$



MCRB – Estimating Sigma

- Yet to be published



Backup

Significance for a Time Scale

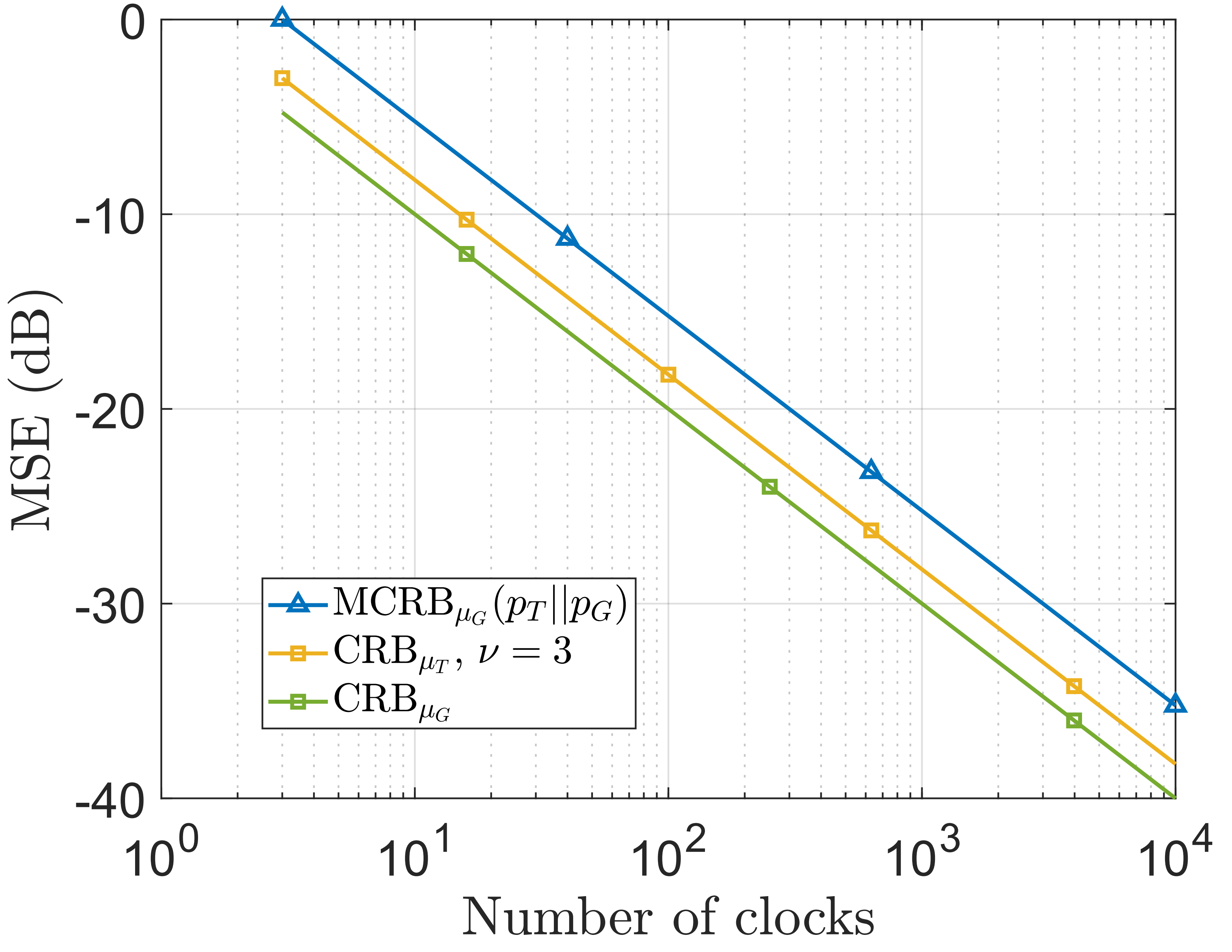
- Time Scale phase is related to estimation error

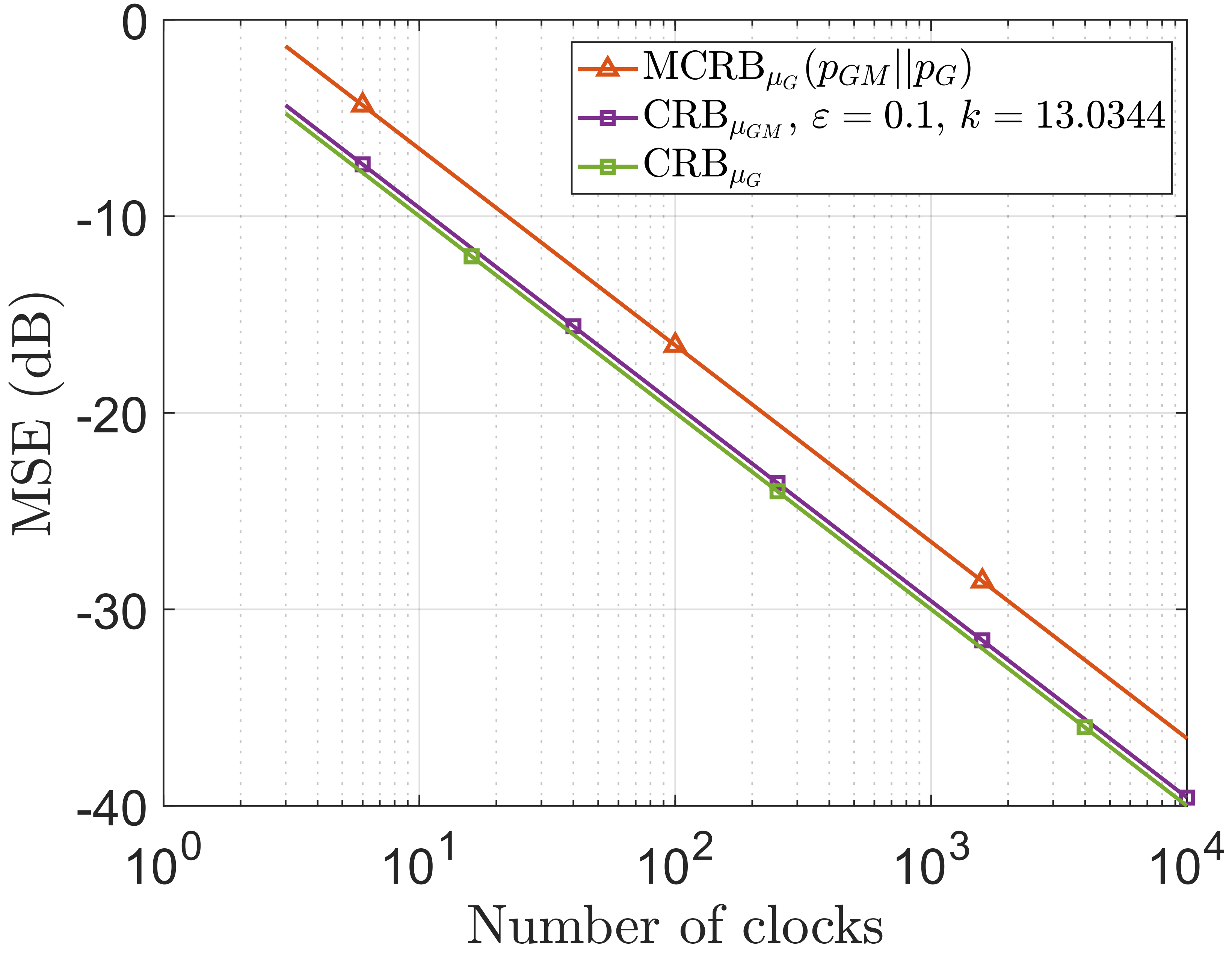
$$E_p[(x_{i,E}(t) - x_{i,p}(t))^2] = E_p[x_{E,p}^2(t)] \geq \mathrm{MCRB}_\mu(t) \geq \mathrm{CRB}_\mu(t)$$

True / Assumed	Gaussian $(t_a - \tau)$	BGM (t_a)	Student's t (t_a)
Gaussian	$\frac{\sigma^2}{N}$	$(\varepsilon(k-1)+1)\frac{\sigma^2}{N}$	$\left(\frac{\nu}{\nu-2}\right)\frac{\sigma^2}{N}$
BGM	$\approx \frac{\sigma^2}{N}$	$\left(\frac{k}{k-\varepsilon(k-1)}\right)\frac{\sigma^2}{N}$	Equivalent $(k, \varepsilon), \nu$
Student's t	$\approx \frac{\sigma^2}{N}$	Equivalent $(k, \varepsilon), \nu$	$\left(\frac{\nu+3}{\nu+1}\right)\frac{\sigma^2}{N}$

Significance for a Time Scale

- Change in bounds relates to possible change in Time Scale phase





Expectation-Maximization: Student's t-distribution

while $\widehat{\boldsymbol{\theta}}_k - \widehat{\boldsymbol{\theta}}_{k-1} > S$

$$u_{j,k}(t) = \frac{\hat{\nu}_{k-1} + 1}{\hat{\nu}_{k-1} + \frac{\left(r_{j,i}(t) - \hat{\mu}_{k-1}(t)\right)^2}{\hat{\sigma}_{k-1}^2}}, \qquad w_{j,k}(t) = \frac{u_{j,k}(t)}{\sum_{l=1}^{N} u_{l,k}(t)},$$

$$\hat{\mu}_k(t) = \sum_{j=1}^{N} w_{j,k}(t) r_{j,i}(t) = x_{i,E}(t)$$

Expectation-Maximization: Student's t-distribution

while $\widehat{\boldsymbol{\theta}}_k - \widehat{\boldsymbol{\theta}}_{k-1} > S$

$$\hat{\sigma}_k^2(t) = \frac{1}{N}\sum_{j=1}^{N} u_{j,k}(t)\left(r_{j,i}(t) - x_{i,E}(t)\right)^2$$

$$\hat{\nu}_k(t), \text{solve:}$$

$$\phi\left(\frac{\hat{\nu}_k(t)}{2}\right) + \sum_{l=1}^{N} u_{j,k}(t) - \psi\left(\frac{\hat{\nu}_{k-1}+1}{2}\right) + \log\left(\frac{\hat{\nu}_{k-1}+1}{2u_{j,k}(t)}\right) - 1 = 0$$

Expectation-Maximization: BGM

while $\widehat{\boldsymbol{\theta}}_l - \widehat{\boldsymbol{\theta}}_{l-1} > S$

$$u_{j,l}(t) = \frac{\hat{\epsilon}_{l-1} g_1\left(r_{j,i}(t); \hat{\mu}_{l-1}, \hat{\sigma}^2_{l-1}, \hat{k}_{l-1}\right)}{(1 - \hat{\epsilon}_{l-1}) g_0\left(r_{j,i}(t); \hat{\mu}_{l-1}, \hat{\sigma}^2_{l-1}\right) + \hat{\epsilon}_{l-1} g_1\left(r_{j,i}(t); \hat{\mu}_{l-1}, \hat{\sigma}^2_{l-1}, \hat{k}_{l-1}\right)},$$

$$g_0\left(r_{j,i}(t); \mu, \sigma^2\right) = \exp\left(-\frac{\left(r_{j,i}(t) - \mu\right)^2}{2\sigma^2}\right), g_1\left(r_{j,i}(t); \mu, \sigma^2, k\right) = \exp\left(-\frac{\left(r_{j,i}(t) - \mu\right)^2}{2k\sigma^2}\right)$$

$$w_{j,l}(t) = 1 - u_{j,l}(t) + \frac{u_{j,l}(t)}{\hat{k}_{l-1}}$$

Expectation-Maximization: BGM

while $\widehat{\boldsymbol{\theta}}_l - \widehat{\boldsymbol{\theta}}_{k-1} > S$

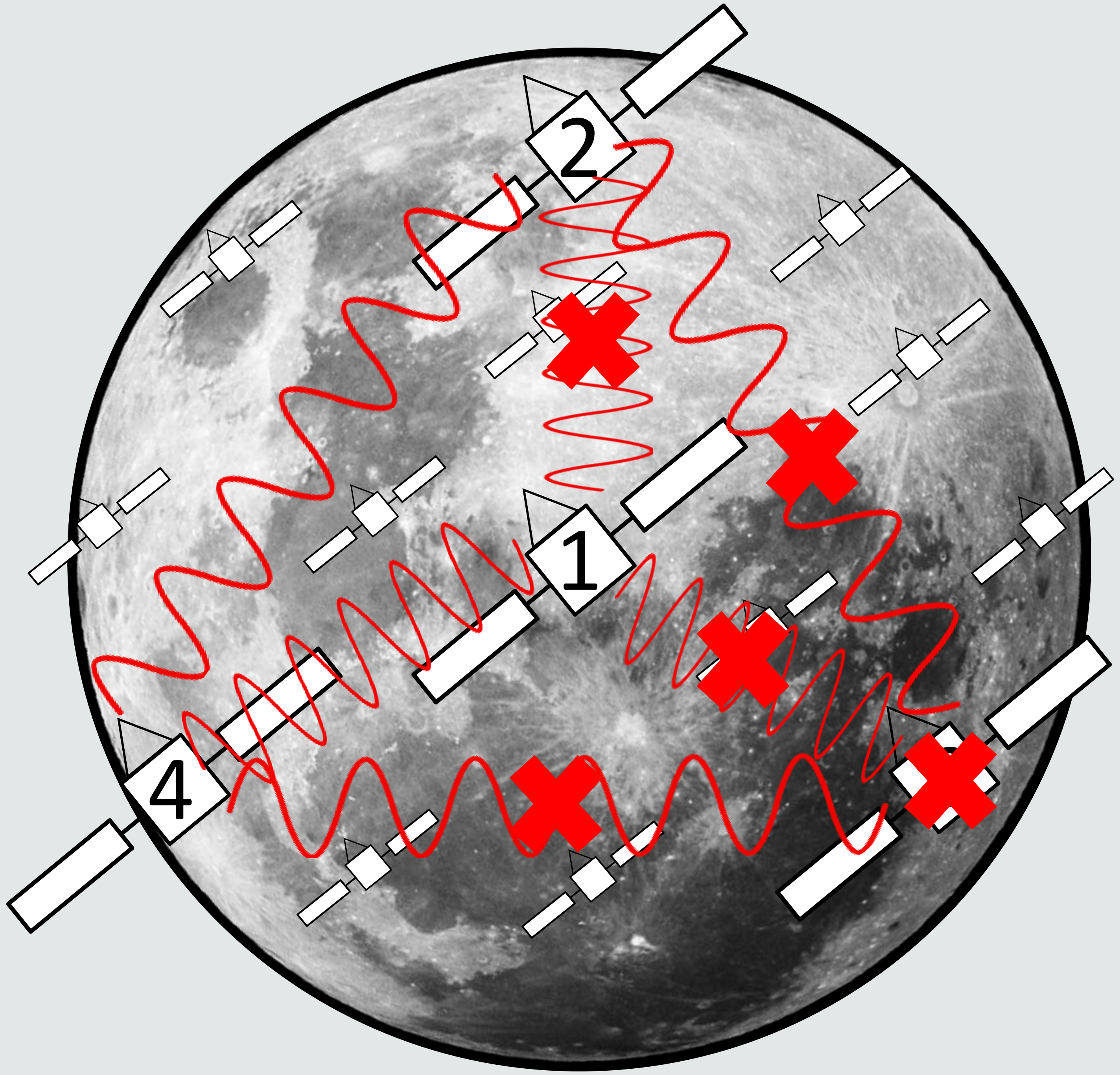
$$\hat{\mu}_l(t) = \frac{\sum_{j=1}^{N} w_{j,l}(t) r_{j,i}(t)}{\sum_{j=1}^{N} w_{j,l}(t)} = x_{i,E}(t)$$

$$\hat{\sigma}_l^2(t) = \frac{1}{N} \sum_{j=1}^{N} w_{j,l}(t) \left(r_{j,i}(t) - \hat{\mu}_l(t) \right)^2$$

$$\hat{\epsilon}_l(t) = \frac{\sum_{j=1}^{N} u_{j,l}(t)}{N}, \qquad \hat{k}_l(t) = \frac{\sum_{j=1}^{N} u_{j,l}(t) \left(r_{j,i}(t) - \hat{\mu}_l(t) \right)^2}{\hat{\sigma}_k^2(t) \sum_{j=1}^{N} u_{j,l}(t)}$$

Treating Missing Links

- Redundant measurements [3]



$$\begin{bmatrix} z_{2,1} \\ z_{3,1} \\ z_{4,1} \\ z_{3,2} \\ z_{4,2} \\ z_{4,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_{2,1} \\ x_{3,1} \\ x_{4,1} \end{bmatrix} + \begin{bmatrix} n_{2,1} \\ n_{3,1} \\ n_{4,1} \\ n_{3,2} \\ n_{4,2} \\ n_{4,3} \end{bmatrix}$$

[3] McPhee, Hamish, et al. 'Exploiting Redundant Measurements for Time Scale Generation in a Swarm of Nanosatellites'. *Proceedings of the 37th Annual European Frequency and Time Forum*, EFTF, Neuchâtel, Switzerland, 2024.

Treating Missing Clocks

- Maintaining continuity in the Time Scale [3]

- N_m clocks lost at time t_m: set weights to zero and renormalize

- N_m clocks returning at time t_r: set weights to zero

- Weights must gradually increase after reintroduction

[3] McPhee, Hamish, et al. 'Exploiting Redundant Measurements for Time Scale Generation in a Swarm of Nanosatellites'. *Proceedings of the 37th Annual European Frequency and Time Forum*, EFTF, Neuchâtel, Switzerland, 2024.

Treating Missing Clocks

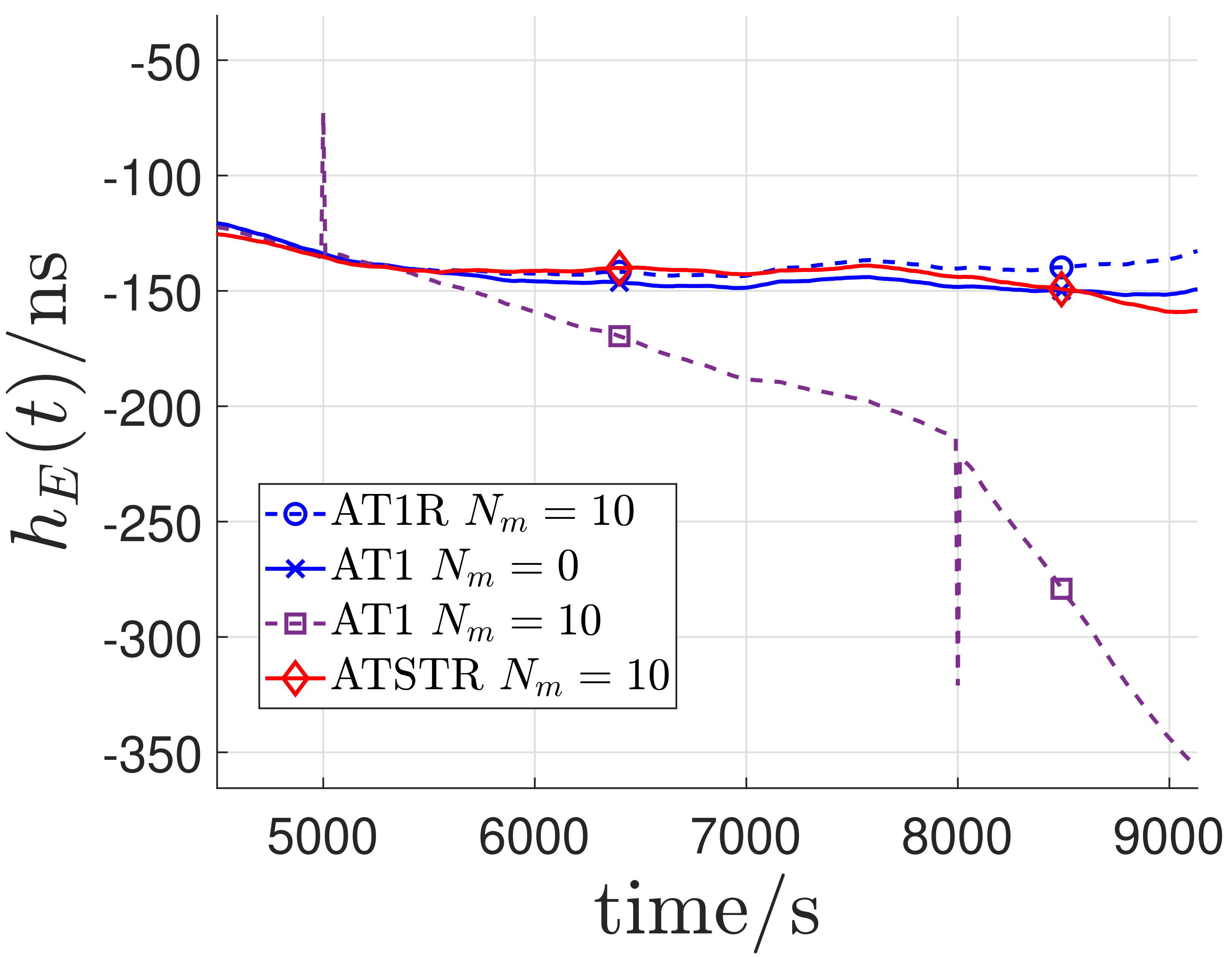
- Maintaining Continuity in the Time Scale [3]

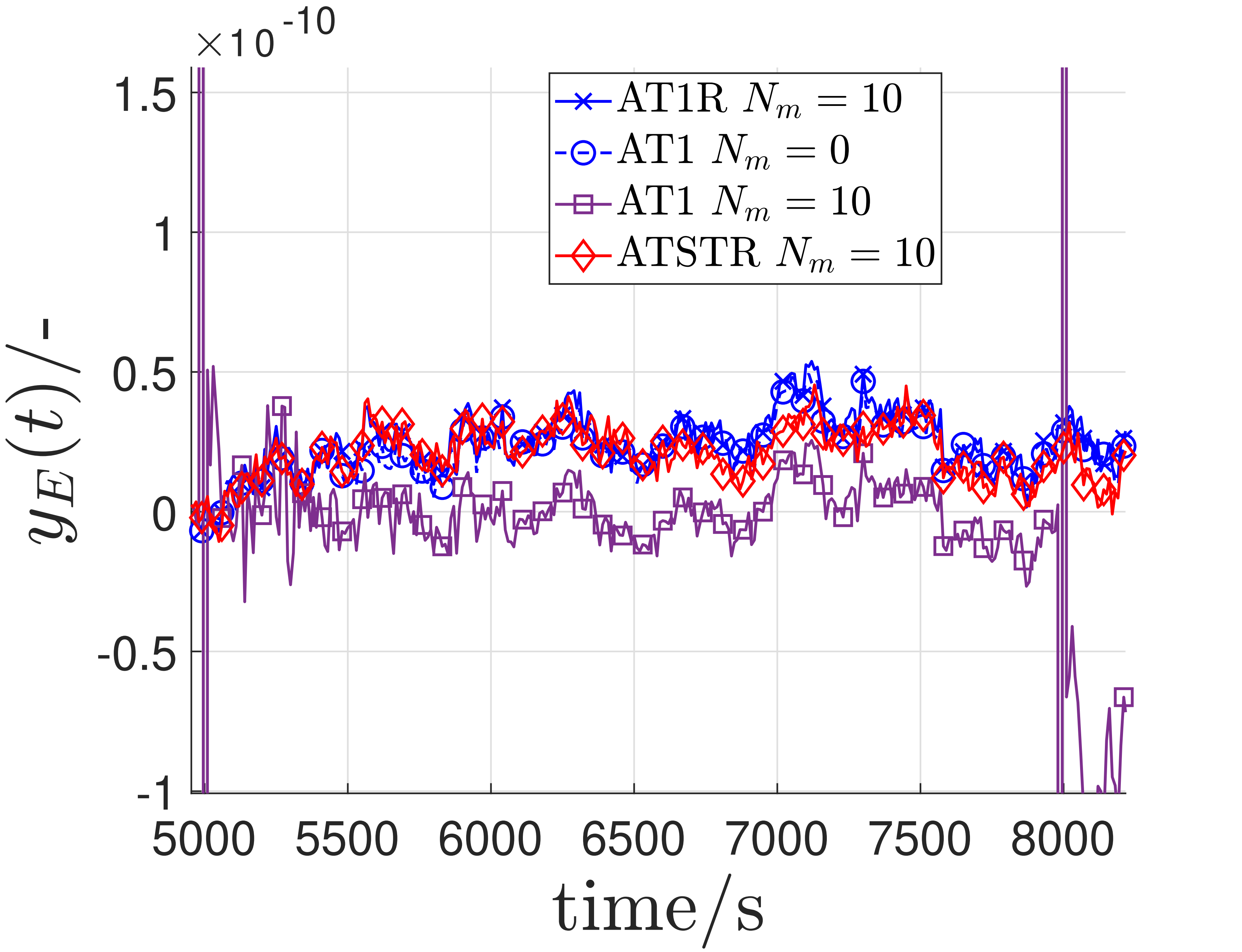
$$C_{N_m}(t_m) = x_{i,E}(t_m)\Big|_{N_m=0} - x_{i,E}(t_m)\Big|_{N_m>0}$$

$$x_{i,E}(t_m)\Big|_{N_a<N} = \sum_{j=1}^{N_a=N-N_m} p_j(t_m-\tau)r_{j,i}(t_m)\,, \qquad p_j(t_m-\tau) = w_j(t_m-\tau)\Big/\sum_{j=1}^{N_a} w_j(t_m-\tau)$$

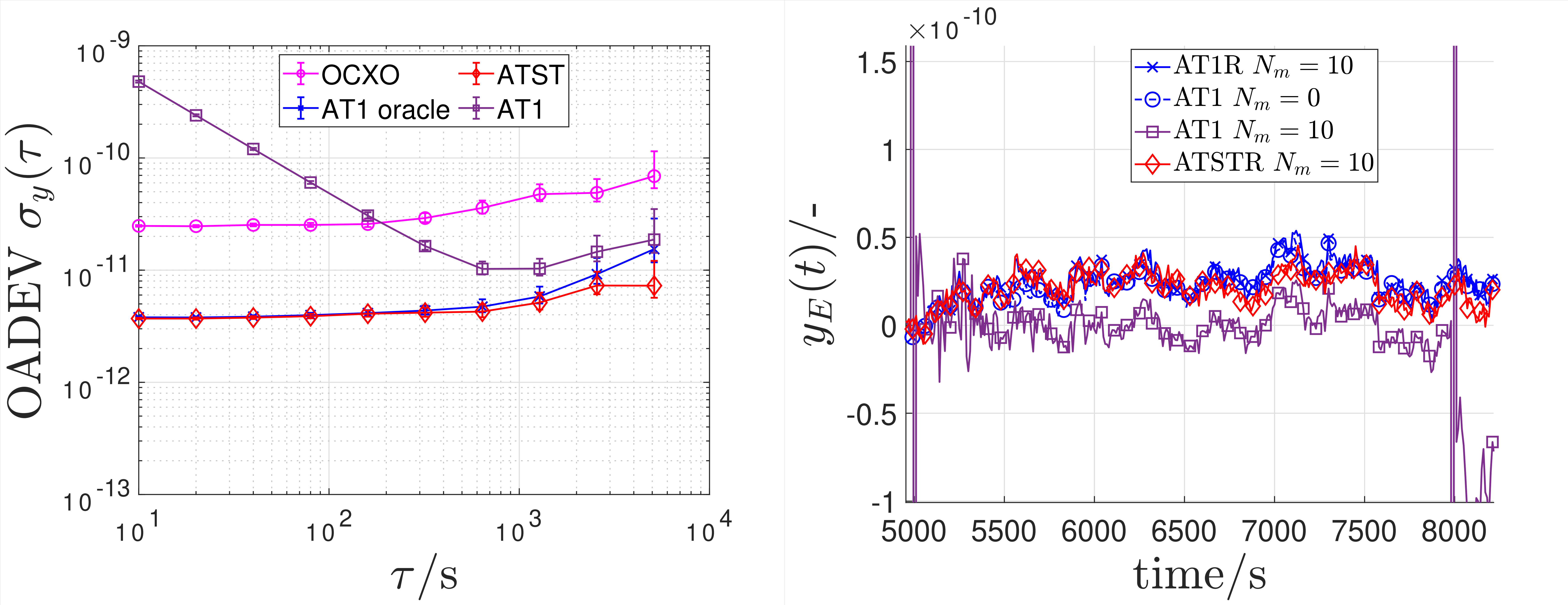
$$C_{N_m}(t_m) = \sum_{j=1}^{N_a}\Big(w_j(t_m-\tau) - p_j(t_m-\tau)\Big)r_{j,i}(t_m) + \sum_{j=N_a+1}^{N} w_j(t_m-\tau)r_{j,i}(t_m)$$

Missing Data

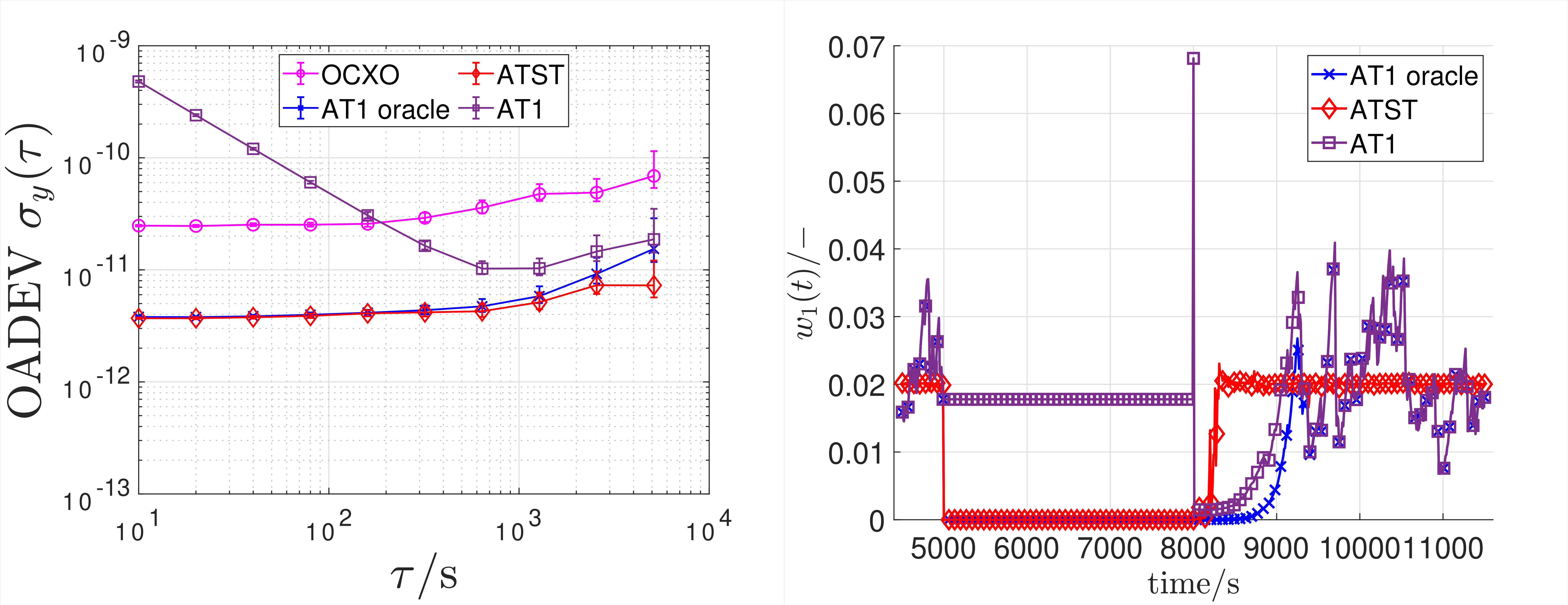




Missing Data



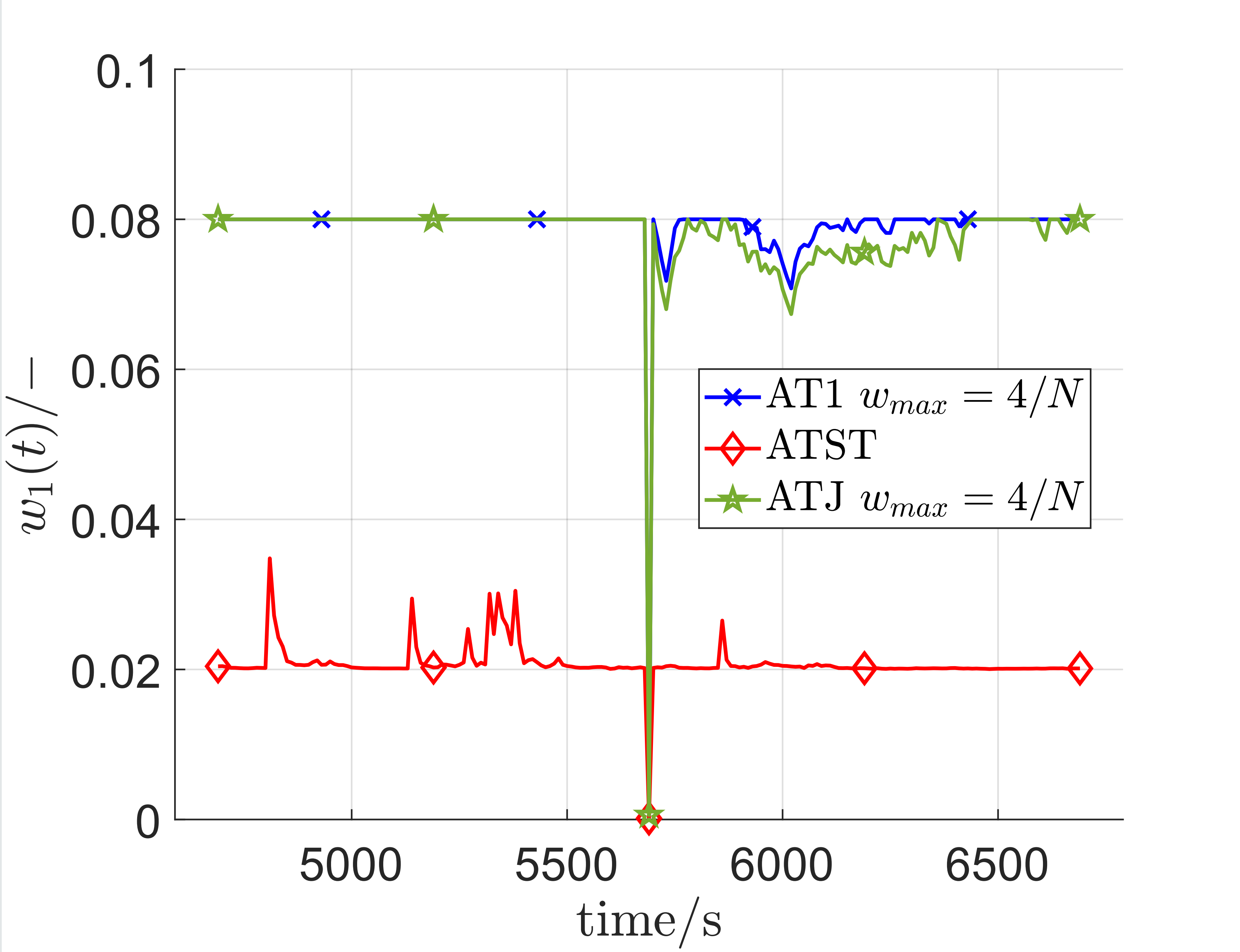
Missing Data



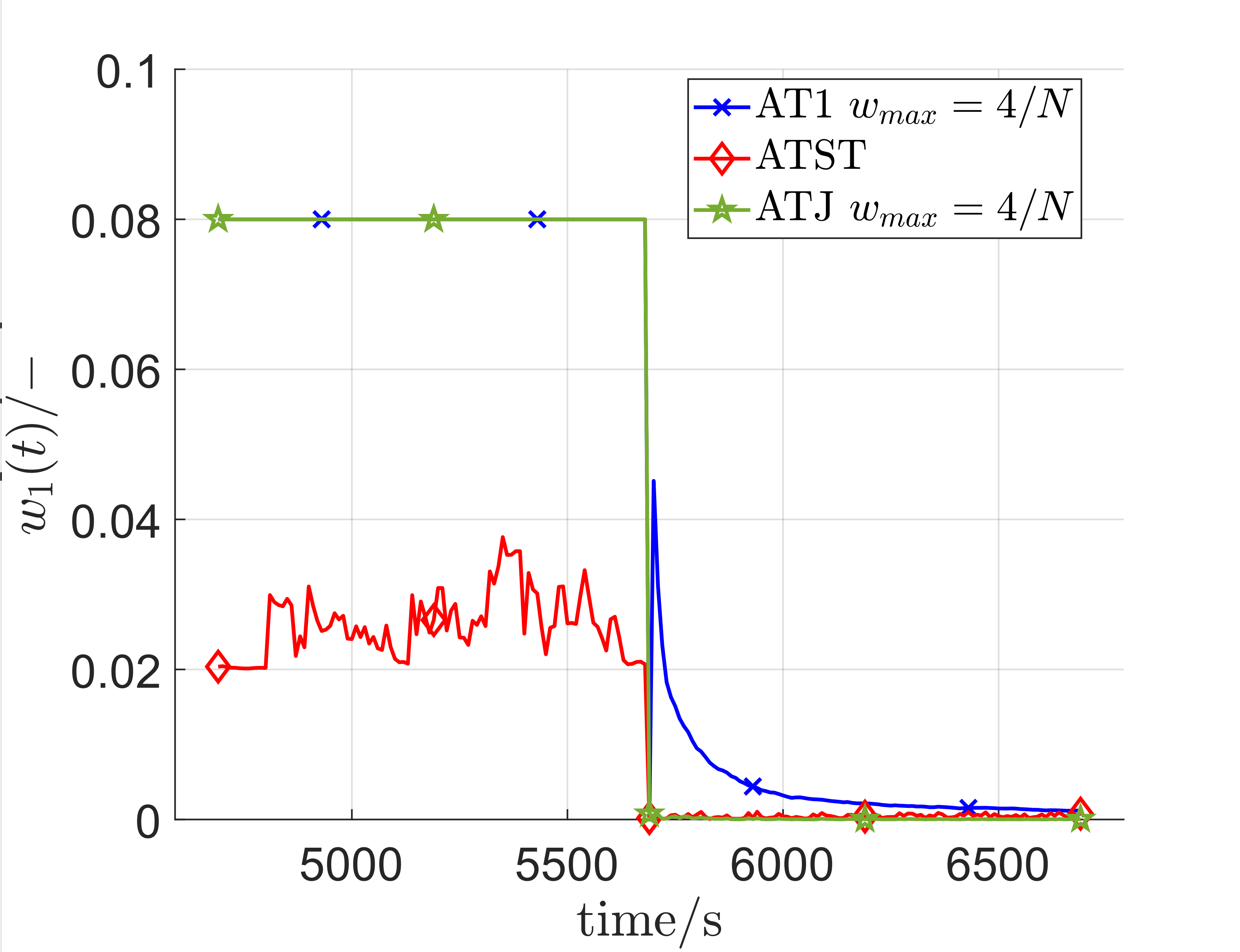
Future work – Joint AT1+ATST

$$w_{j,\mathrm{ATJ}}(t) = \frac{\frac{w_{j,\mathrm{ATST}}(t)}{\epsilon_j^2(t)}}{\sum_{i=1}^{N} \frac{w_{i,\mathrm{ATST}}(t)}{\epsilon_j^2(t)}}$$

Future work – Joint AT1+ATST



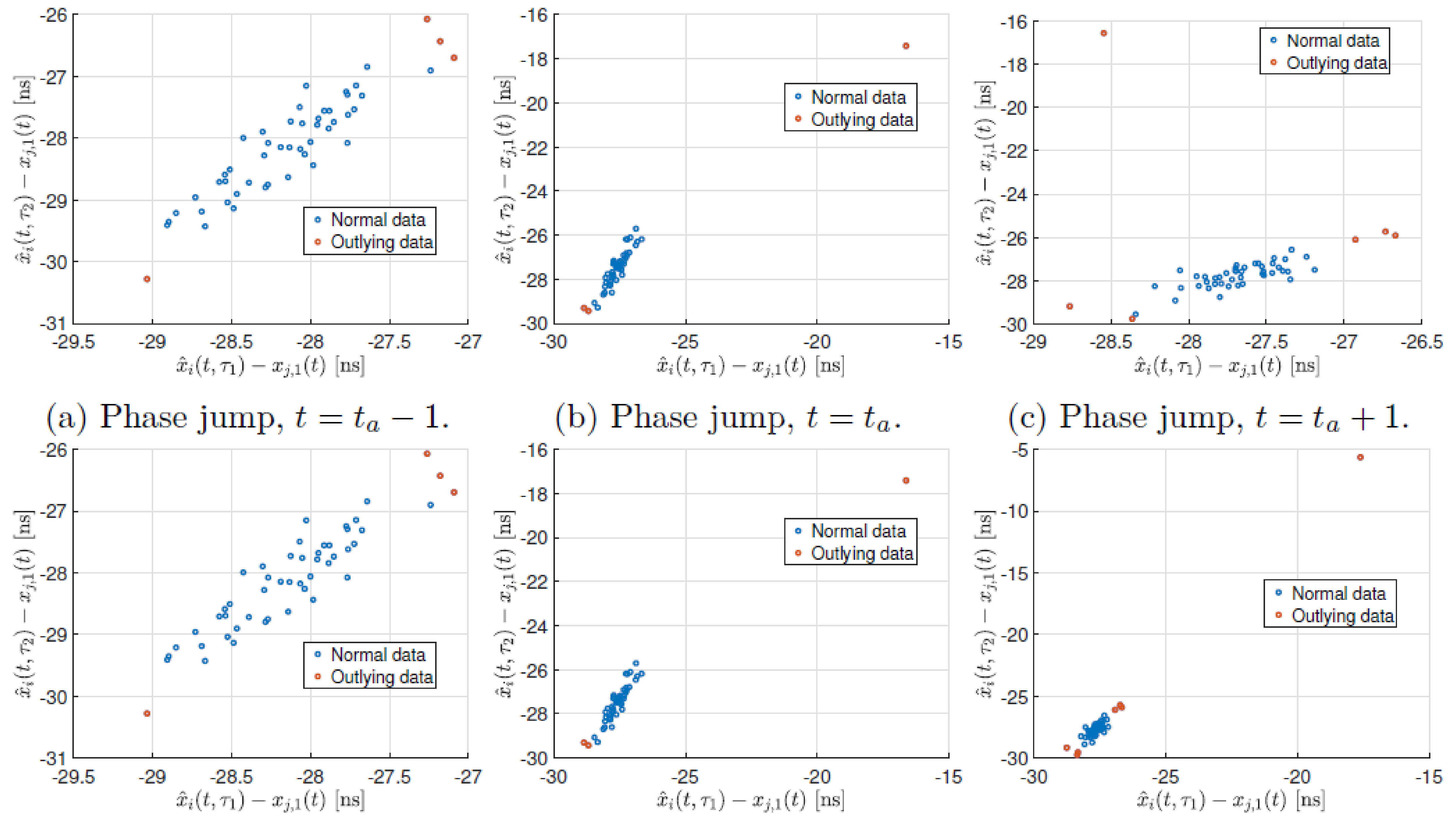
Phase Jump



Frequency Jump

Future work – Machine Learning

- Outlier Scores
  - Detection
  - Weights
- Isolation Forest
- Local Outlier Factor
- And more



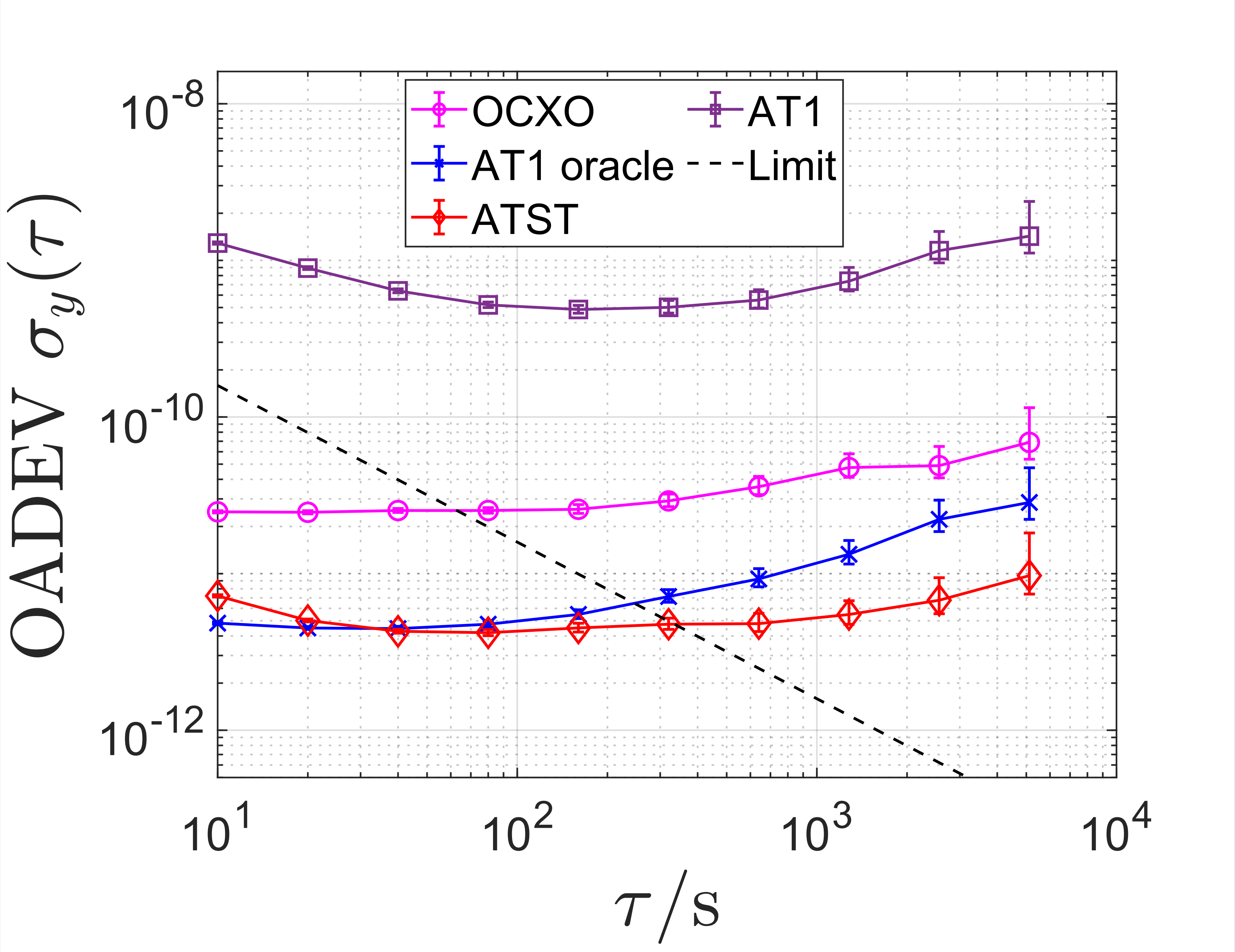
(a) Phase jump, $t = t_a - 1$. (b) Phase jump, $t = t_a$. (c) Phase jump, $t = t_a + 1$.

(d) Frequency jump, $t = t_a - 1$. (e) Frequency jump, $t = t_a$. (f) Frequency jump, $t = t_a + 1$.

Future work – Transient Anomalies

Examples

- Temporary Frequency Jumps

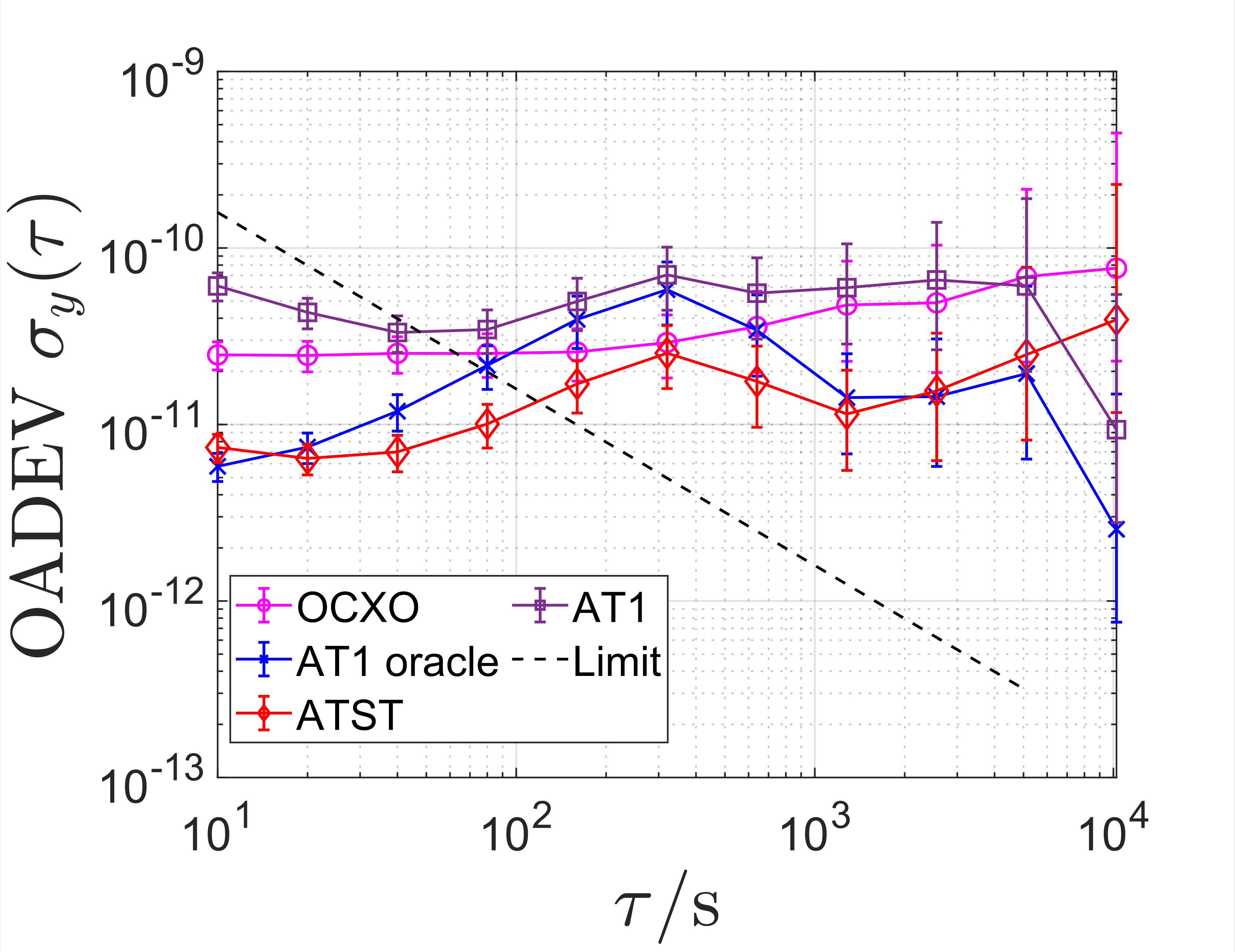


Future work – Transient Anomalies

Examples

- Temporary Frequency Jumps

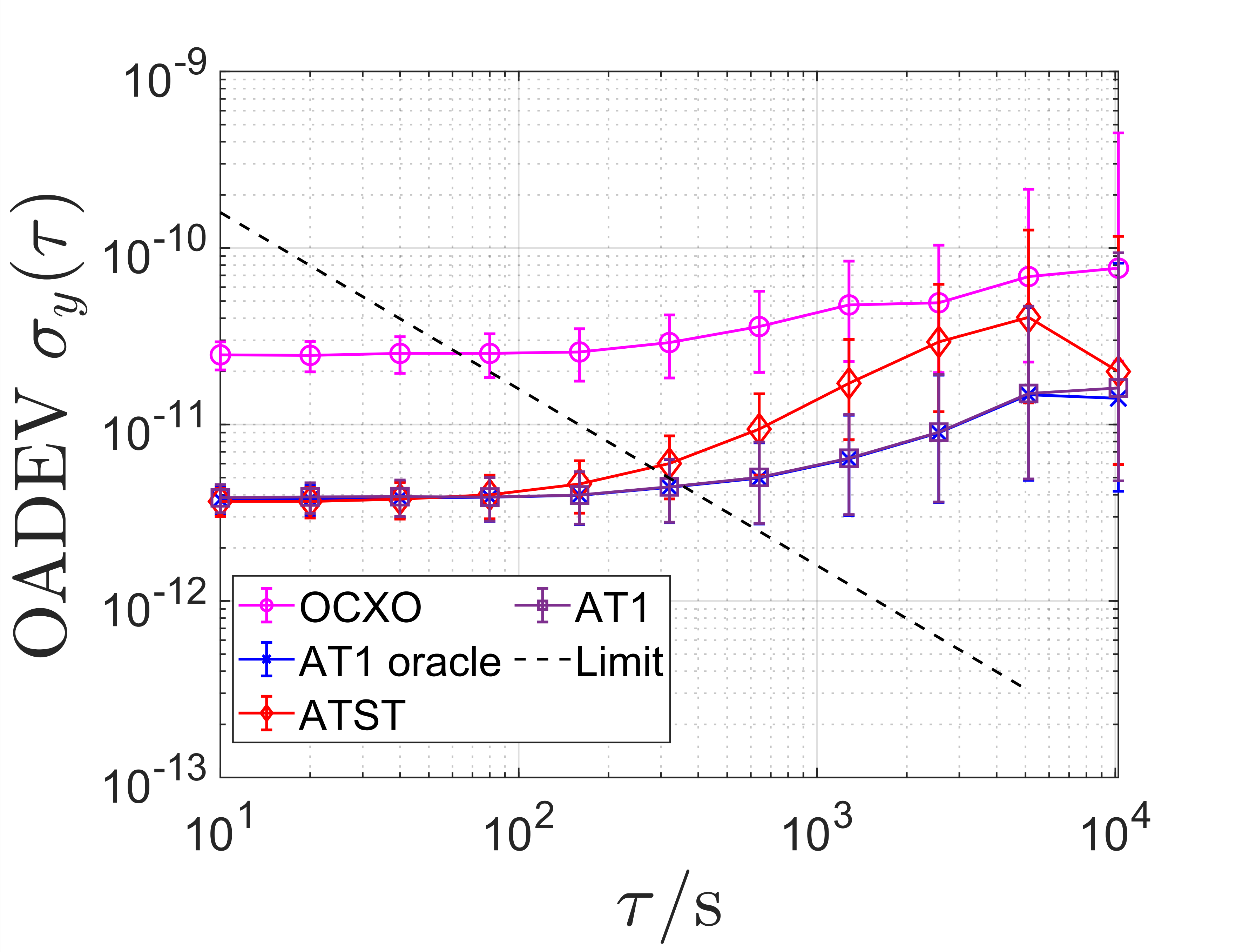
- Periodic effects



Future work – Transient Anomalies

Examples

- Temporary Frequency Jumps
- Periodic effects
- Frequency Drift



Future work – Transient Anomalies

Solutions?

- Machine Learning
- Robust Frequency Estimation